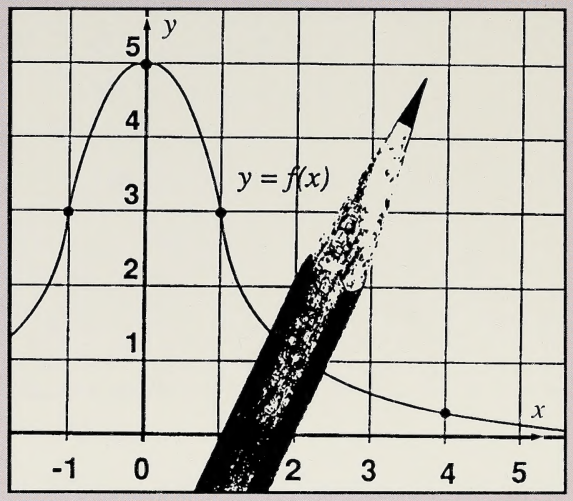
MATHEMATICS 3

Functions and Relations



Unit 3

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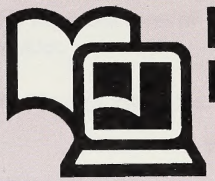

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 33 Student Module Unit 3 Functions and Relations Alberta Distance Learning Centre ISBN No. 0-7741-0139-3

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General Information

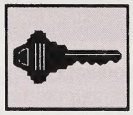
This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.
- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

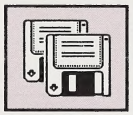


Key Idea
- flagging important ideas

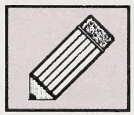
Audiotape
- learning by listening to an audiotape

What You Already Know
- reviewing what you already know

Another View
- exploring different perspectives



Computer Software
- learning by using computer software



Review
- studying previous concepts

Solutions
- correcting the activities

Videotape
- learning by viewing a videotape

Introduction
- introducing the unit

Extra Help
- providing additional study

Print Pathway
- choosing a print alternative

What Lies Ahead
- previewing the unit

Extensions
- going on with the topic

Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts

What You Have Learned
- summarizing what you have learned

Mathematics 33

Course Overview

Mathematics 33 contains 8 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Powers and Radicals	10%
Unit 2 Polynomials and Rational Expressions	10%
Unit 3 Functions and Relations	16%
Unit 4 Quadratic Functions and Equations	20%
Unit 5 Trigonometry	16%
Unit 6 Statistics	16%
Unit 7 Annuities	6%
Unit 8 Mortgages and Loans	6%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Functions and Relations

This unit covers topics dealing with functions and relations. Each topic contains explanations, examples, and activities to assist you in understanding functions and relations. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in **Appendix A**. In several cases there is more than one way to do the question.

Unit 3 Functions and Relations

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	• What Lies Ahead	• Extensions	
	• Exploring Topic 1		
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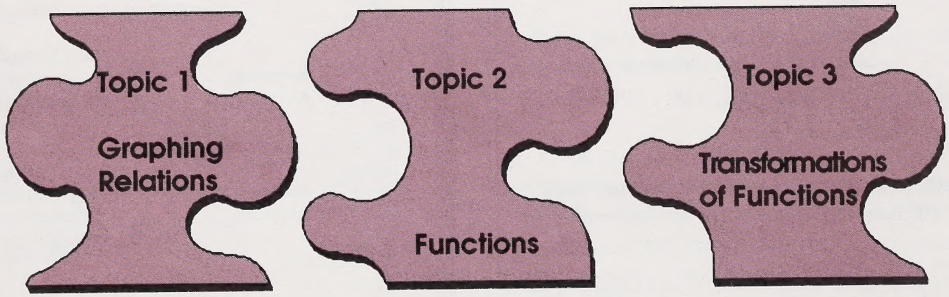
Functions and Relations

People are often concerned about their relationships with other people. Family relationships, friendly relationships, and business relationships are a few examples of such relationships.

Numbers and sets of numbers are often related. The salaries people earn are often related to their job classifications and/or the number of years of experience. The cost of goods is related to the cost of production.

In this unit you will study a variety of relations and a special form of relation called a **function**.

Unit 3 Functions and Relations



What You Already Know

Refresh your memory!

You have learned how to draw the graphs of direct, partial, and inverse variations.

Example 1

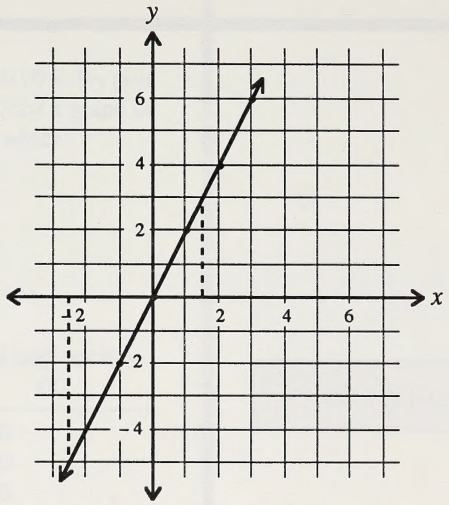
- Draw the graph of $y = 2x$.

Solution:

Make a table of values:

x	y
-1	-2
0	0
1	2
2	4
3	6

Draw the graph using the values in the table.



This is an example of **direct variation.**

The **dependent variable** is y.
The **independent variable** is x.

- From the graph determine the value of y when $x = 1.5$.

Solution:

Draw a vertical line at $x = 1.5$ (the dotted line on the graph), and read the value of y where this line intersects the graph.
When $x = 1.5$, $y = 3$.

This process is called **interpolation.**

- From the graph determine the value of y when $x = -2.5$.

Solution:

Draw a vertical line at $x = -2.5$ (the dotted line on the graph), and read the value of y when this line intersects the graph. (You may have to extend the graph first.)
When $x = -2.5$, $y = -5$.

This process is called **extrapolation.**

Example 2

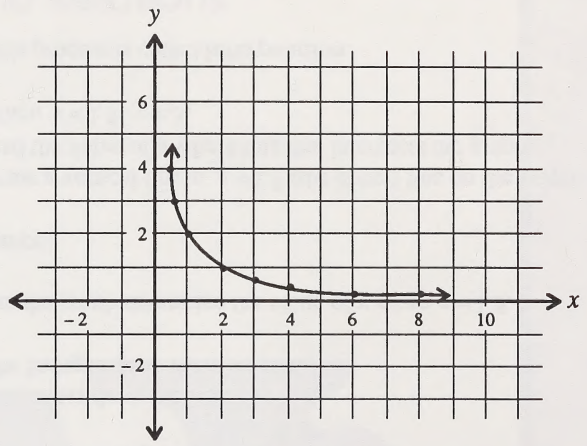
Draw the graph of $y = \frac{2}{x}$.

Solution:

Make a table of values. Notice that x cannot equal zero since division by zero is not allowed.

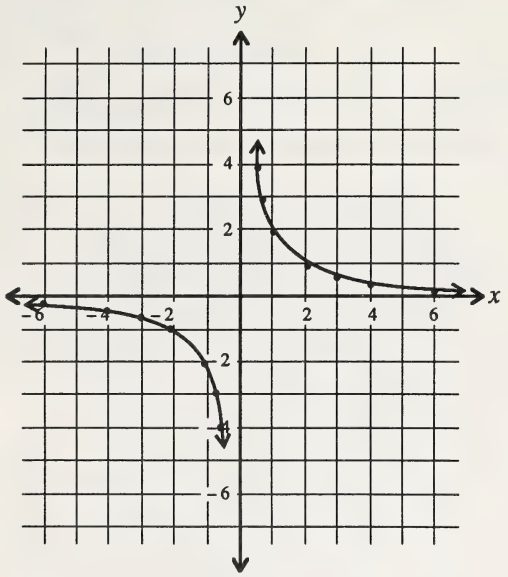
x	y
1	2
2	1
3	$\frac{2}{3}$
4	$\frac{1}{2}$
6	$\frac{1}{3}$
8	$\frac{1}{4}$

Draw the graph using the values in the table.



This is an example of **inverse variation.**

The preceding example shows the graph when x is positive. You could also draw the graph including negative x-values. You would get the following graph.



Example 3

A car rental company charges \$30.00 a day plus ten cents per kilometre driven. Draw a graph of this relation. Write the equation which represents this relation.

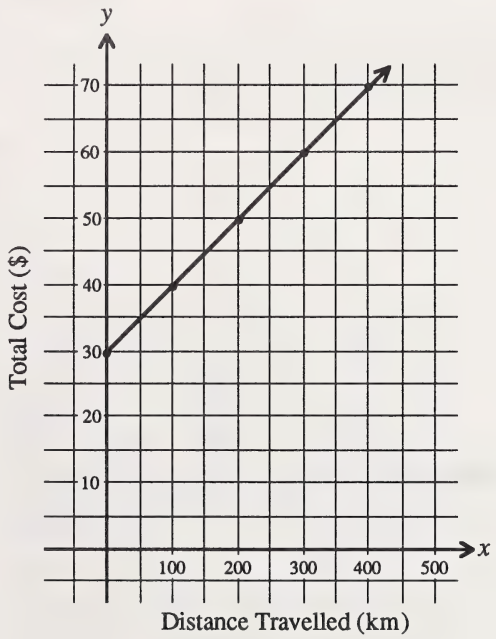
Solution:

Make a table of values.

Total driven in one day (km)	Total cost per day (\$)
0	30.00
100	40.00
200	50.00
300	60.00

(\$0.10)(100) = \$10.00

Draw the graph using the values in the table.



The equation which represents this relation is as follows:

$$C = \$30.00 + (\$0.10)n$$

This is an example of **partial variation.**

You have learned how to graph linear relations given a certain amount of information.

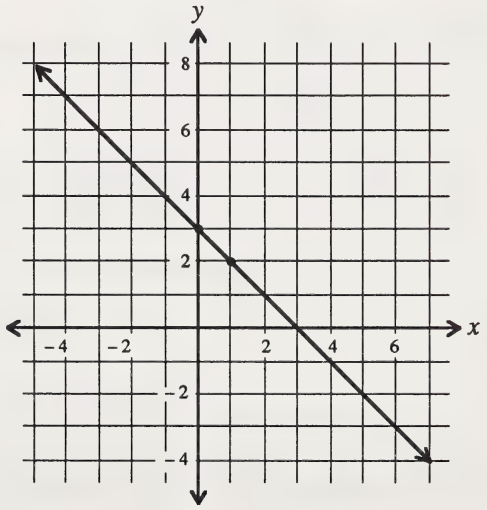
Example 4

Draw the graph of a linear relation with slope -1 and y-intercept 3.

Solution:

The y-intercept 3 means the point $(0, 3)$.

Plot the point $(0, 3)$ and find a second point which produces a slope of -1 from the y-intercept. Join the two points with a straight line.



$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

A slope of -1 means moving one unit down and one unit to the right.

C = total cost ($)

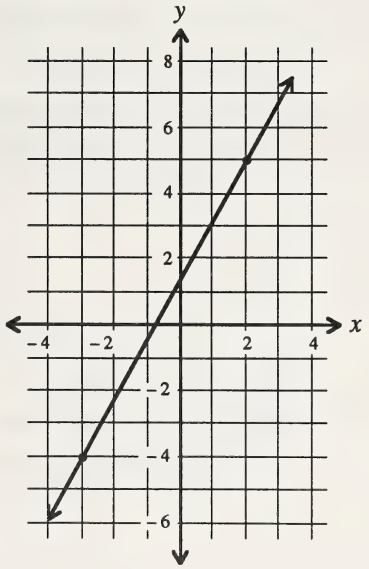
n = number of kilometres driven

Example 5

Draw the graph of a linear relation when $(-3,-4)$ and $(2,5)$ are on the line.

Solution:

Plot the points on the graph and join them with a straight line.

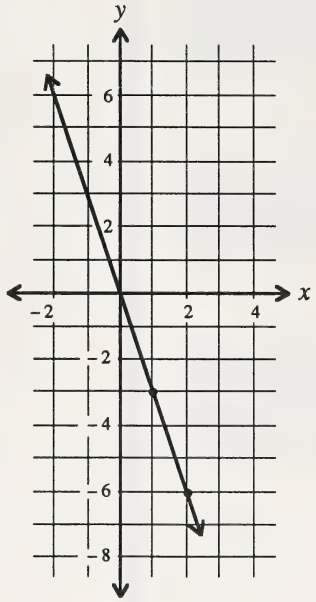


Example 6

Draw the graph of a linear relation that passes through $(2,-6)$ and has a slope of -3.

Solution:

Plot the point on the graph. From that point select the point three units up and one unit to the left. Then join the points with a straight line.

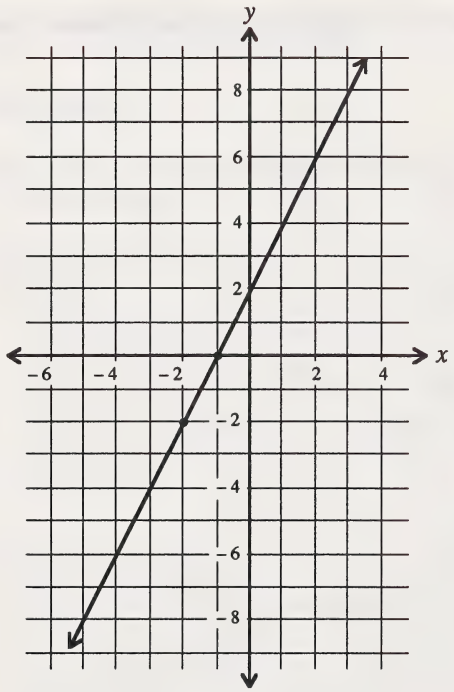


Example 7

Draw the graph of the linear relation which passes through the point $(-2,-2)$ and is parallel to the line $y=2x+5$.

Solution:

Determine the slope of $y=2x+5$. The slope is 2. Plot the point $(-2,-2)$ on the graph paper. Find a second point two units up and one unit to the right. Draw a straight line through these two points.



Note: $y=mx+b$, where m is equal to the slope.

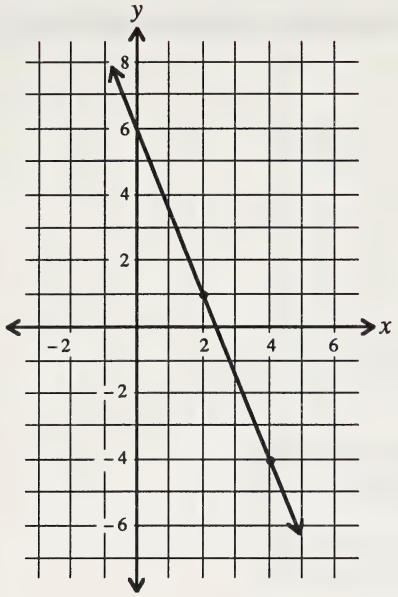
Slopes of parallel lines are equal.

Example 8

Draw the graph of a linear relation which passes through the point $(4, -4)$ and is perpendicular to the line $y = \frac{2}{5}x + 3$.

Solution:

Find the slope of the line $y = \frac{2}{5}x + 3$. The slope is $\frac{2}{5}$. Find the negative reciprocal of the slope. The negative reciprocal of $\frac{2}{5}$ is $-\frac{5}{2}$. Plot the point $(4, -4)$. From this point go up five units and to the left two units. Mark this point and join both points with a straight line.

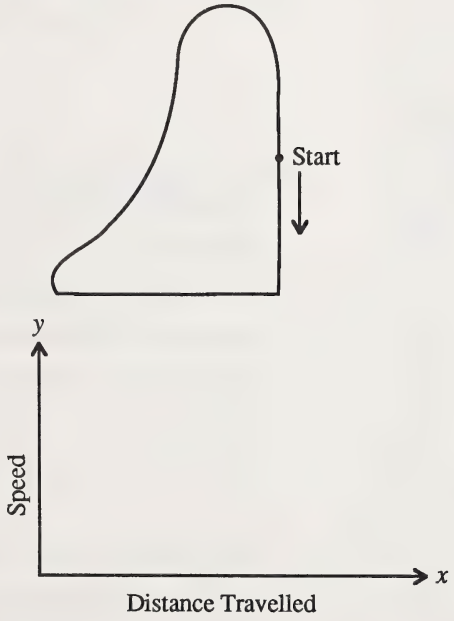


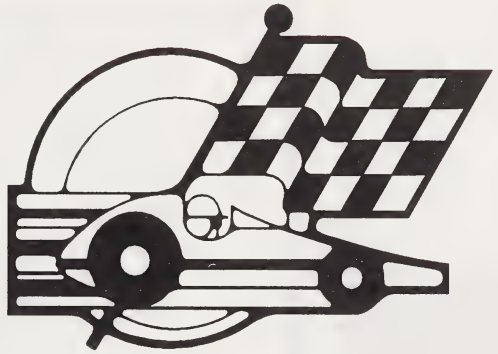
Slopes of perpendicular lines are negative reciprocals of each other.

Review

Do any three of questions 1 to 5. Then do any four of questions 6 to 11. Graph paper is provided in **Appendix B.**

1. You are a driver of a Formula 3 race car in the Red Deer 500. Sketch a graph of speed versus distance for a lap around the track from a standing start.





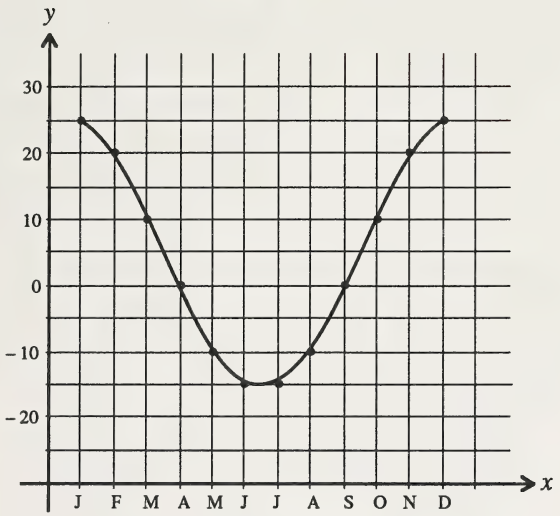
2. Draw the graph of $y = \frac{1}{2}x + 2$. State the type of relation represented: direct, inverse, or partial variation.

3. Draw the graph of $y = 5x$. State the type of relation represented.

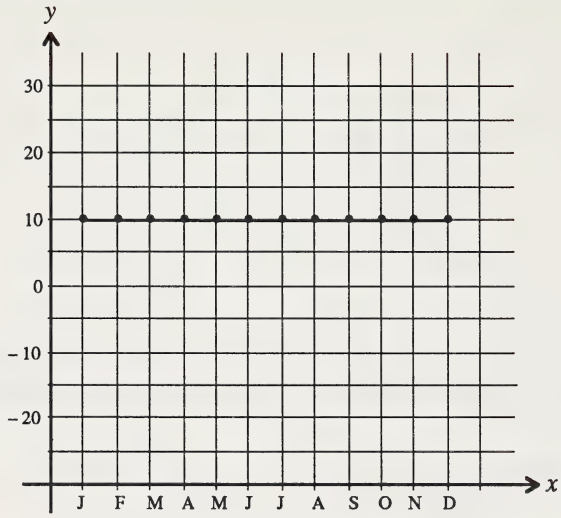
4. Draw the graph of $xy = 2$. State the type of relation represented.

5. Select the graph that best represents the average temperature for Edmonton, Alberta. Explain the reason for your selection.

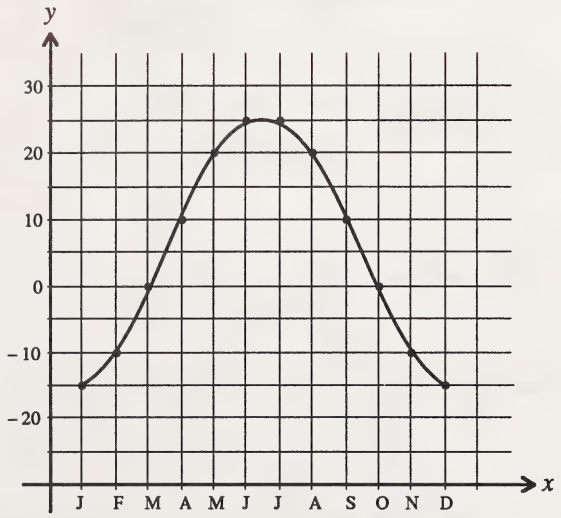
A.



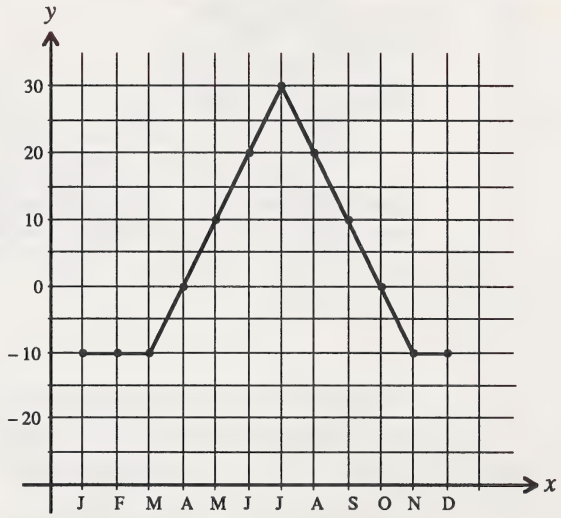
B.



C.

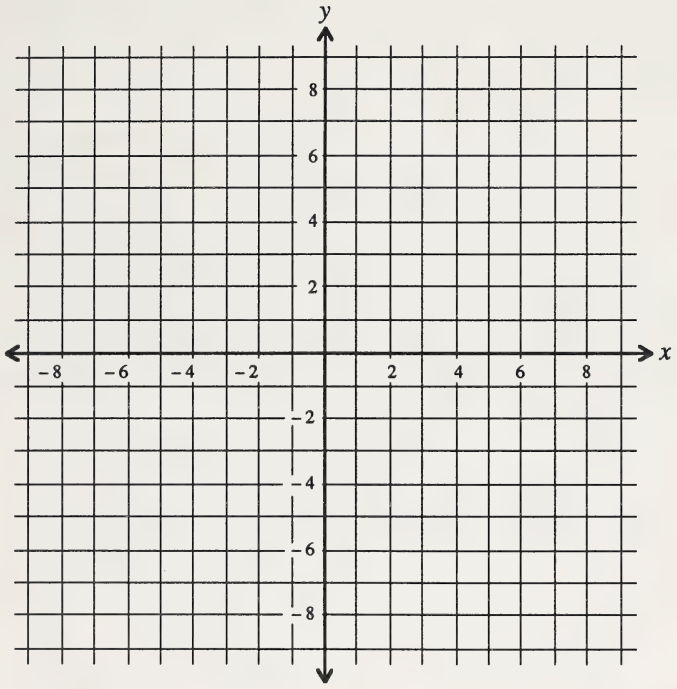


D.

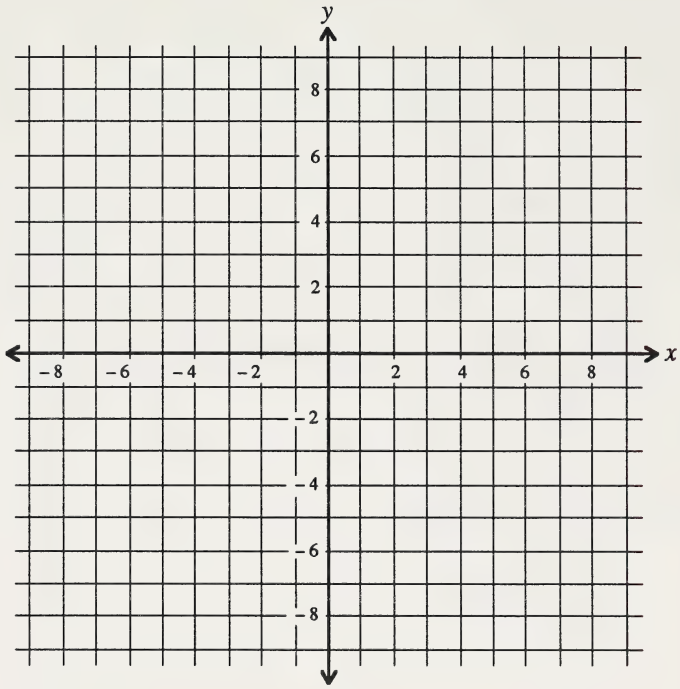


If you had problems with questions 1 to 5, please review **Mathematics 13, Unit 5**.

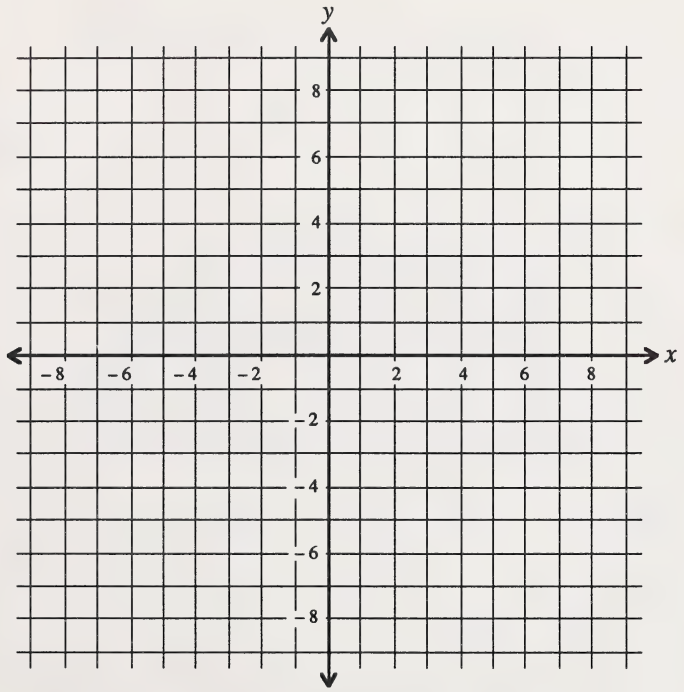
6. Draw the graph of the line with a slope of 6 and the y-intercept -2.



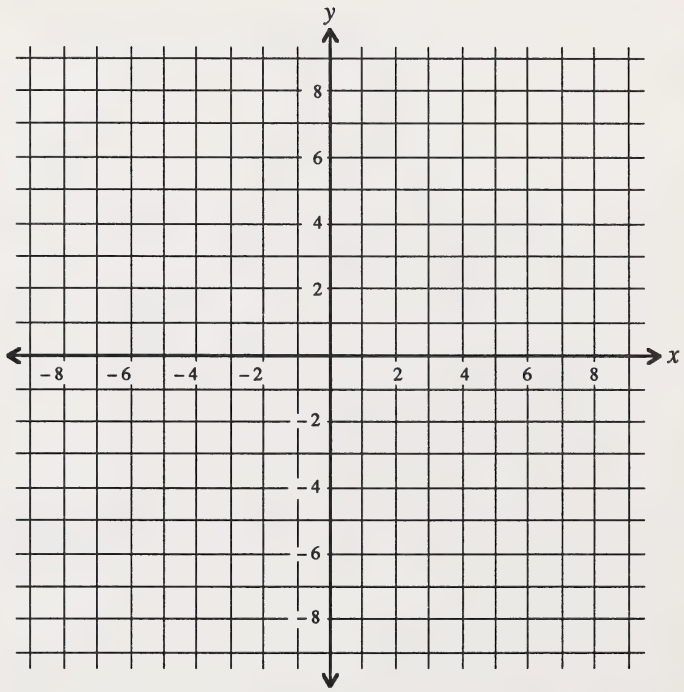
7. Draw the graph of the linear relation passing through the points $(-4, 3)$ and $(6, -5)$.



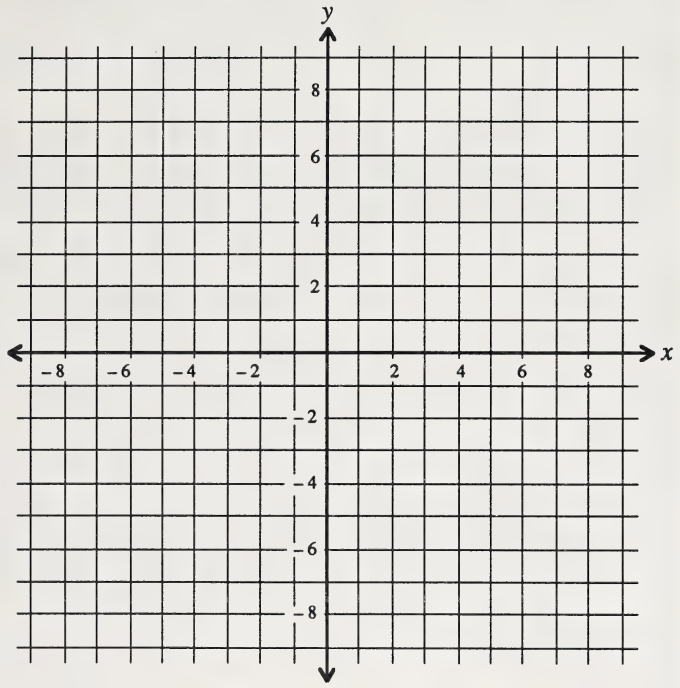
8. Draw the graph of the linear relation which is perpendicular to the line $y = -\frac{2}{3}x - 9$ and passes through the point $(-6, -3)$.



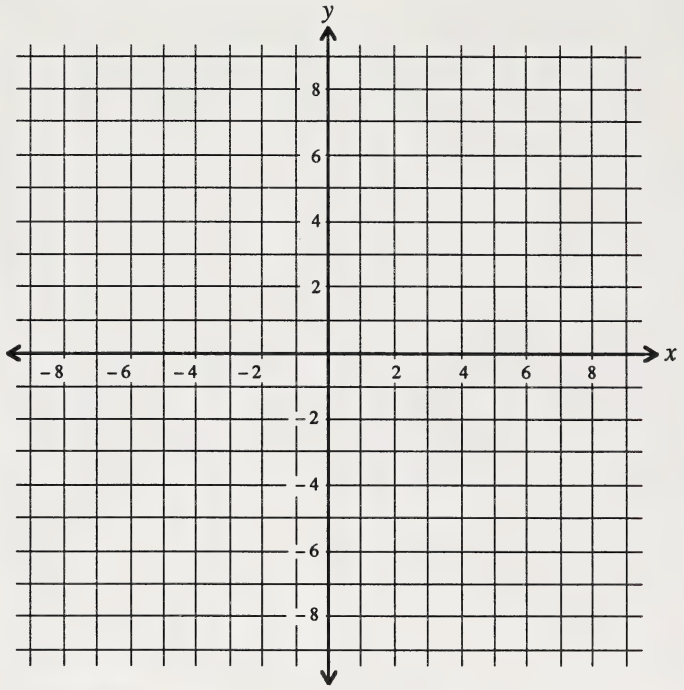
9. Draw the graph of the line which passes through $(0, 0)$ and is parallel to the line $6x + 2y = 5$.



10. Draw the graph of the line which passes through $(4, 6)$ and has a slope of $-\frac{7}{3}$.



11. Draw the graph of the linear relation which has a y-intercept equal to -5 and is parallel to the line $y = \frac{1}{2}x - 3$.



If you had problems with questions 6 to 11, please review **Mathematics 23, Unit 4**.

Now go to the **Review** solutions in **Appendix A**.

Topic 1 Graphing Relations

Introduction

When a person gains or loses weight, what factor is most responsible? Usually the factor most responsible is the number of Calories consumed per day. The more high energy foods you eat, the more weight you gain. This common relationship between food and weight is the basis for many diet schemes and athletic training programs.

To better understand these and other everyday occurrences, a solid background in relations is required.

If you have access to a videocassette recorder (VCR), you may wish to view the video titled **Introduction to Relations**. This video is program 1 of the *Math Sense*[1] series and will introduce you to the basic concepts of a relation.

[1] *Math Sense* is a title of ACCESS Network.

What Lies Ahead

Throughout the topic you will learn to

1. graph relations that describe everyday occurrences and physical phenomena

2. solve problems by graphing and interpreting the graphs which describe everyday occurrences and physical phenomena

Now that you know what to expect, turn the page to begin your study of graphing relations.

Exploring Topic 1

Activity 1

Graph relations that describe everyday occurrences and physical phenomena.

How are people related? Normal close relations are described by parent, son, daughter, sister, brother, mother-in-law, and so on. These terms refer to the relationship one person has to another. People can also be related geographically, nationally, religiously, culturally, and so on. As long as some rule or definition exists between two people, they are related to each other by that rule or definition.

Variables can also be related. When variables are related, they are related mathematically. This leads to the definition of a relation.

A mathematical **relation** is a set of ordered pairs of real numbers.

Examine some relations and their graphs.

Example 1

Some people will spend most of their money on chocolate bars. If chocolate bars cost 75¢ ($0.75) each, determine the relationship between the number of chocolate bars purchased and the amount of money spent (total cost).

Solution:

Here is a table in which the number of chocolate bars purchased is given with the total cost of the chocolate bars.

Number of chocolate bars purchased	Total cost of chocolate bars ($)
0	0.00
1	0.75
2	1.50
3	2.25
4	3.00
5	3.75
6	4.50
7	5.25

This table can be expressed as a set of ordered pairs. The number of chocolate bars purchased is the first number and the total cost of the chocolate bars is the second number.

The set of ordered pairs is as follows:

$(0, 0.00)$

$(1, 0.75)$

$(2, 1.50)$

$(3, 2.25)$

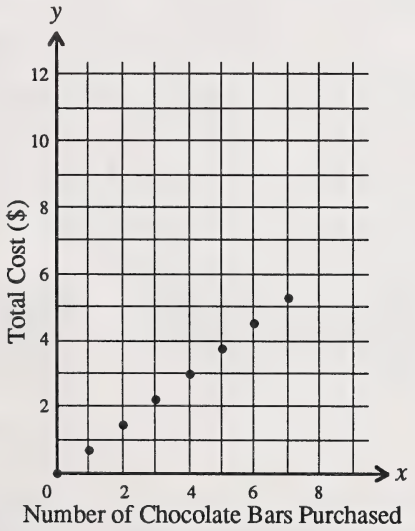
$(4, 3.00)$

$(5, 3.75)$

$(6, 4.50)$

$(7, 5.25)$

This set of ordered pairs can be plotted on a graph.



Notice that the points on the graph are **not** joined by a line. Why not? Well, how many times have you purchased $\frac{1}{2}$ a chocolate bar in the store?

This relation can be expressed as a sentence.

The total cost in dollars is equal to \$0.75 times the number of chocolate bars purchased.

This relation can be expressed as a mathematical equation.

$$C = (\$0.75)n;\ n \in W$$

Notice the restrictions on the values of n. Since chocolate bars come in whole number amounts, n must be a whole number.

There are a number of generalizations about relations that can be drawn from this example. Relations can be expressed as the following:

- table of values
- set of ordered pairs
- graph
- sentence
- mathematical equation

C = total cost
n = number of chocolate bars purchased
W = the set of whole numbers

Another example of an everyday occurrence is a taxi ride.

Example 2

A simplified taxi fare scheme goes as follows: You pay a fixed start-up charge and then an additional distance charge. (In reality there is also a time charge.) If the start-up charge is \$2.00 and the distance charge is 10¢ per 0.25 km, then determine the relationship between total cost and distance travelled.

Solution:

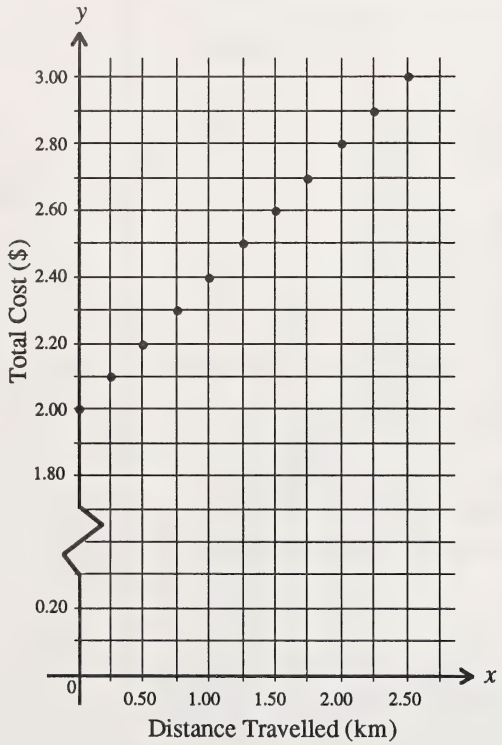
The following data can be calculated:

Distance travelled (km)	Total cost (\$)
0.00	2.00
0.25	2.10
0.50	2.20
0.75	2.30
1.00	2.40
1.25	2.50
1.50	2.60
1.75	2.70
2.00	2.80
2.25	2.90
2.50	3.00

The set of ordered pairs is as follows:

$(0.00, 2.00)$
$(0.25, 2.10)$
$(0.50, 2.20)$
$(0.75, 2.30)$
$(1.00, 2.40)$
$(1.25, 2.50)$
$(1.50, 2.60)$
$(1.75, 2.70)$
$(2.00, 2.80)$
$(2.25, 2.90)$
$(2.50, 3.00)$

The graph of the data is as follows:



This relation can be expressed as a sentence.

The cost of a taxi ride is equal to \$2.00 plus the variable cost of \$0.10 per 0.25 km travelled.

This relation can be expressed as an equation. You will use the following variables:

$C =$ total cost in dollars

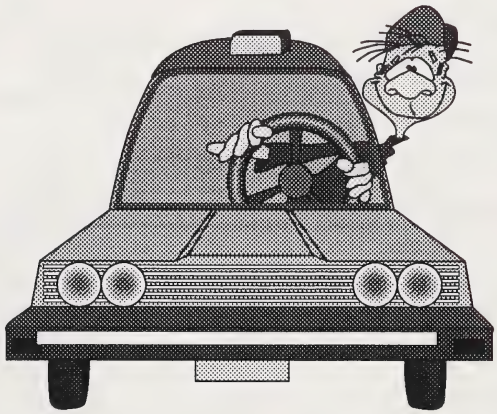
$F =$ fixed start - up charge in dollars

$d =$ distance travelled in kilometres

$$C = F + \frac{0.10d}{0.25}$$

$$C = 2.00 + 0.40d$$

where $d = 0.00$ km, 0.25 km, 0.50 km, 0.75 km, 1.00 km, . . .



Notice that d has to be in increments of 0.25 km.

This symbol indicates a break from the ordered increments in the scale. The graph would be too large if all values of the total cost were included from \$0.00 to \$3.00.

Example 3

The time it takes to travel a certain distance is related to the average speed. Consider the distance from the outskirts of Edmonton to Barrhead to be 100 km.

Given certain average speeds, use the physics formula $t = \frac{d}{v}$ to calculate the time required to make the trip.

Solution:

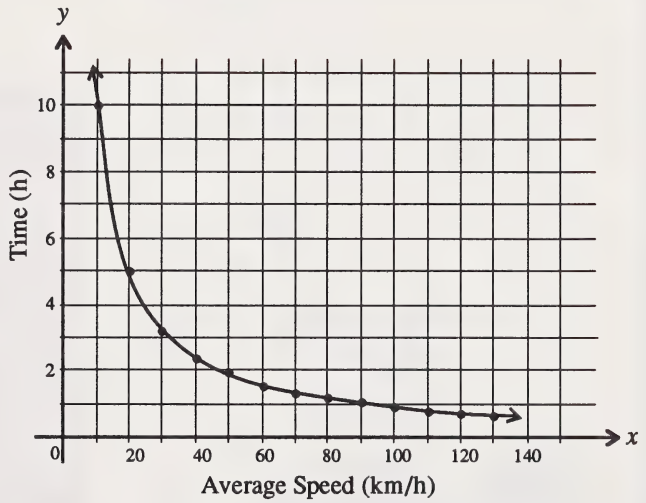
Average speed (km/h)	Time to make the trip (h)
10	10.00
20	5.00
30	3.33
40	2.50
50	2.00
60	1.67
70	1.43
80	1.25
90	1.11
100	1.00
110	0.91
120	0.83
130	0.77

The set of ordered pairs is as follows:

(10, 10.00)
(20, 5.00)
(30, 3.33)
(40, 2.50)
(50, 2.00)
(60, 1.67)
(70, 1.43)
(80, 1.25)
(90, 1.11)
(100, 1.00)
(110, 0.91)
(120, 0.83)
(130, 0.77)

$$\text{time} = \frac{\text{distance}}{\text{average speed}}$$

The graph of the data is as follows:



This relation can be expressed as a sentence.

The time required to drive from Edmonton to Barrhead is equal to 100 km divided by the average speed travelled.

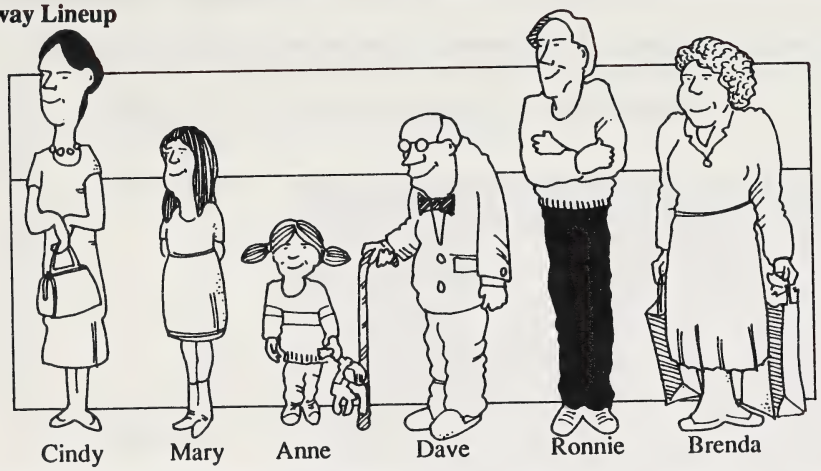
As an equation, write $t = \frac{100}{v_{avg}}$, where t = time in hours and

v_{avg} = average speed in kilometres per hour.

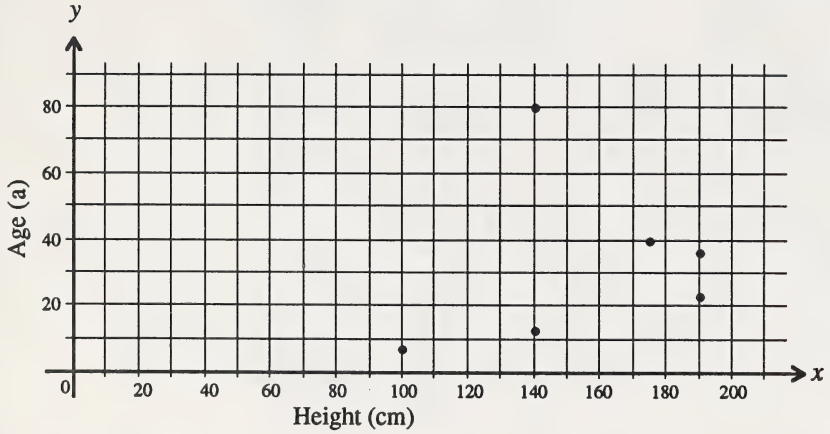
Why are the points on the graph joined with a smooth curve? Average speed is a continuous quantity here. A person could drive at any speed within the limits of the vehicle (and the law). A vehicle has an entire range of possible speeds (up to the maximum possible). Therefore, the time for the trip is a continuous curve which depends only on the average speed.

Example 4

The Subway Lineup

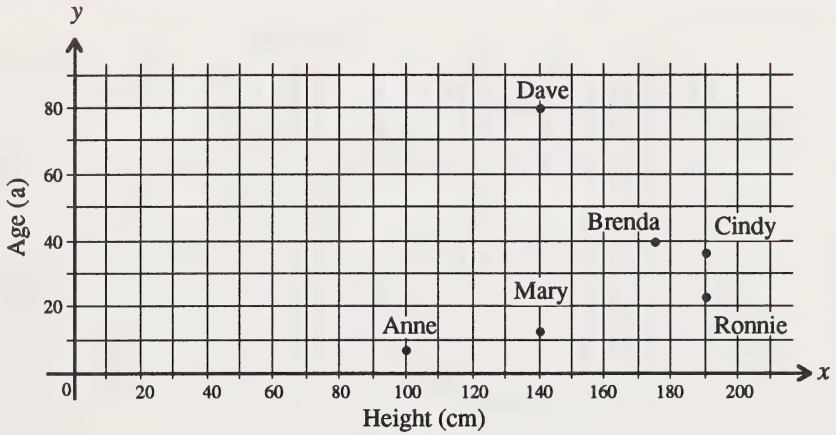


Each person in the subway lineup is represented by a point on the graph. Write the name of the person beside the point representing that person.



The abbreviated form for year is a.

Solution:



This relationship can be represented by a table.

Height (cm)	Age (a)	
100	8	(Anne)
140	12	(Mary)
140	80	(Dave)
175	40	(Brenda)
190	23	(Ronnie)
190	38	(Cindy)

This relationship can be expressed as a set of ordered pairs.

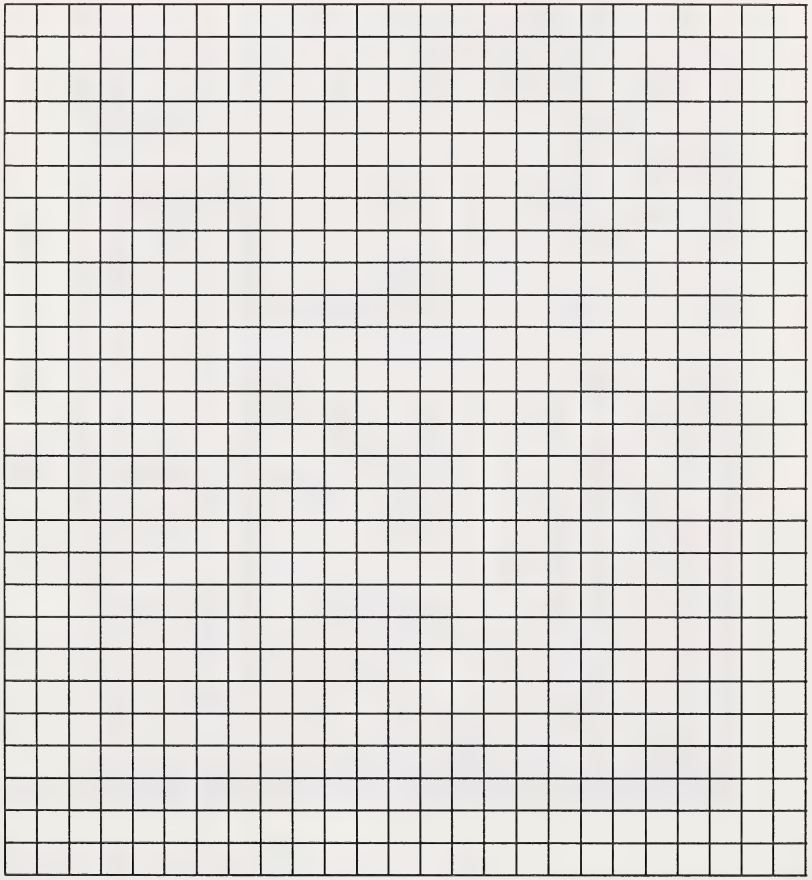
$(100, 8), (140, 12), (140, 80), (175, 40), (190, 23), (190, 38)$

There is no equation that can be written for this mathematical relationship.

Do any three of the following questions.

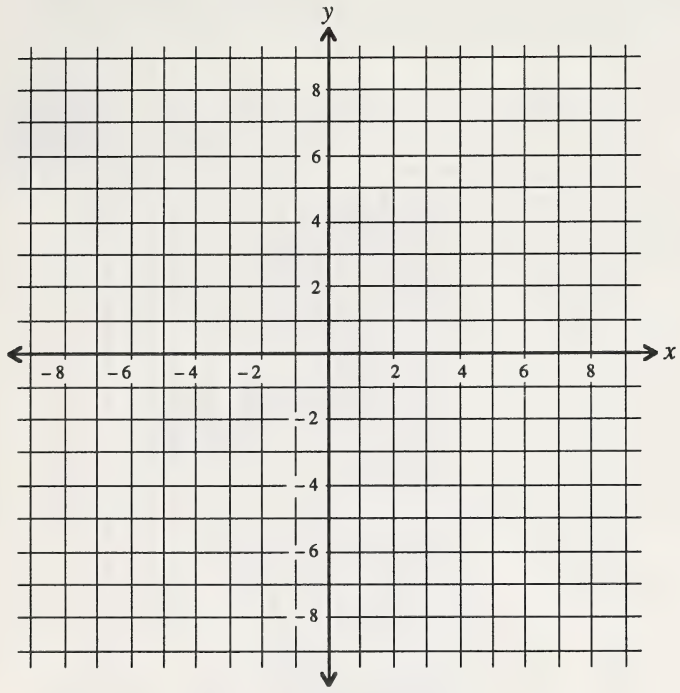
1. Draw a graph of the relationship between the number of hours of sunlight per day and the height of a bean plant in centimetres after one week. (The plants were initially 10 cm tall.) The first entry would show that this group of plants received one hour of sunlight per day for one week in a laboratory setting. A blank grid is given following the table.

Sunlight per day (h)	Height of a bean plant (cm)
1	10.1
2	10.2
3	10.4
4	10.6
5	10.9
6	11.2
7	11.6
8	12.1
9	12.7
10	13.4
11	14.2
12	15.1
13	16.2
14	17.4
15	19.1
16	21.0
17	22.7
18	22.0
19	21.5
20	21.0
21	20.5
22	20.0
23	19.5
24	19.0



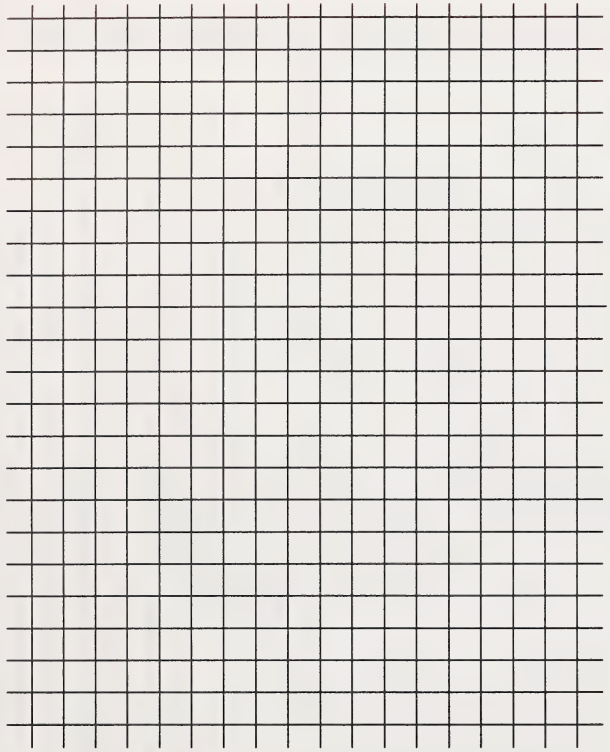
2. Draw the graph of the following ordered pairs. Label each ordered pair.

$(1,6)$, $(5,2)$, $(-1,5)$, $(-6,2)$, $(-4,-3)$, $(-2,-6)$, $(2,-5)$, $(6,-6)$, $(0,0)$



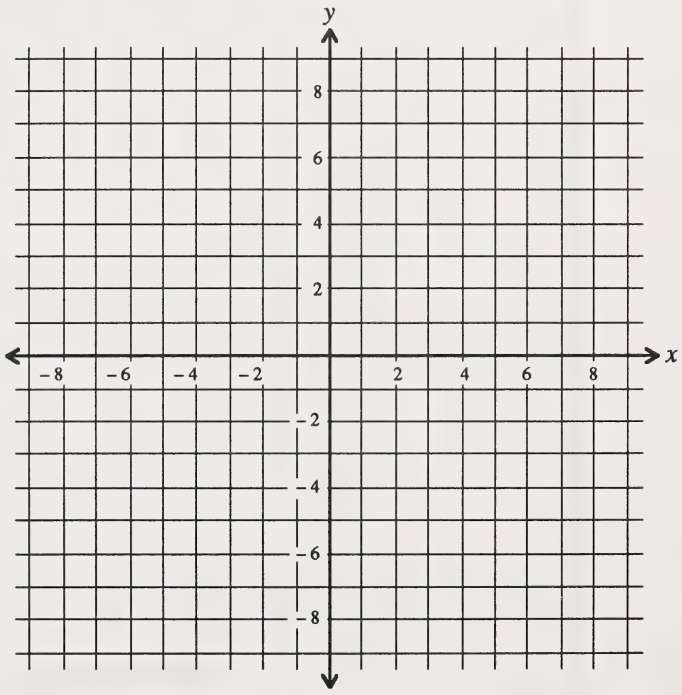
3. The amount of fuel required to accelerate a heavy four-wheel-drive vehicle doubles for every increase of 20 km/h beyond 40 km/h. If 0.20 L of fuel is required to accelerate the vehicle to 40 km/h, draw a graph of the fuel required to accelerate the vehicle to 140 km/h. A blank grid is given following the table.

Speed (km/h)	Fuel consumed (L)



4. Draw the graph of the equation $y = -x^2 + 8$.
   Use values of x such that x is an element of the set of real numbers.

x	
y	



5. The weight gain of an average person is related to the number of Calories consumed per day. The average male requires 3600 Calories per day to maintain his body weight. He gains or loses one kilogram for the consumption of each 7300 Calories more or less than this required amount. Draw the graph of the relationship between Calories consumed each day and weight change when an average male consumes a range of 1000 to 6000 Calories per day. Graph paper is provided in **Appendix B.**

For solutions to **Activity 1**, turn to **Appendix A, Topic 1**.

Activity 2

Solve problems by graphing and interpreting the graphs which describe everyday occurrences and physical phenomena.

Everyday problems can sometimes be very difficult to solve. Often a problem can be viewed from a different perspective. This can simplify the problem. One of these methods is to graph the problem, and then interpret the graph to solve the problem.

Example 5

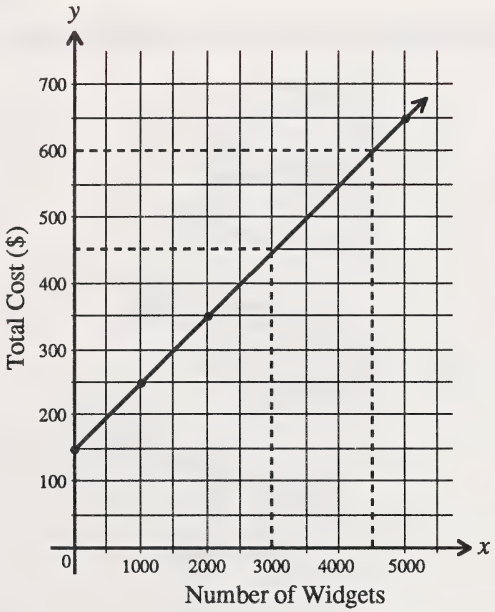
A machine that makes widgets costs a company $150 per day to rent plus $0.10 per widget produced.

- Draw the graph of the number of widgets produced versus the total cost per day of producing the widgets.

Solution:

Make a table of values.

Number of widgets produced	0	100	200	1000	2000	5000
Total cost per day	$150	$160	$170	$250	$350	$650



- What is the cost when the machine produces 3000 widgets?

  Solution:

  From the graph the cost to produce 3000 widgets is $450.

- How many widgets could be produced per day for a fixed cost of $600 per day for the machine?

  Solution:

  From the graph 4500 widgets could be produced for $600.

Example 6

The Freely Falling Object

An object falls from an airplane flying at an altitude of 2500 m. The relationship between the distance the object falls and the time taken to fall that distance is given by $d = \frac{1}{2}at^2$, where

d = distance in metres

$a = 9.8 \text{ m/s}^2$ (acceleration due to gravity)

t = time in seconds

- Draw the distance-time graph for this relation.

  Solution:

  First make a table of values for various times.

Time (s)	Distance fallen (m)
0.0	0.0
1.0	4.9
2.0	19.6
3.0	44.1
4.0	78.4
5.0	122.5
10.0	490.0
15.0	1102.5
20.0	1960.0
22.6	2500.0

Some sample calculations are as follows:

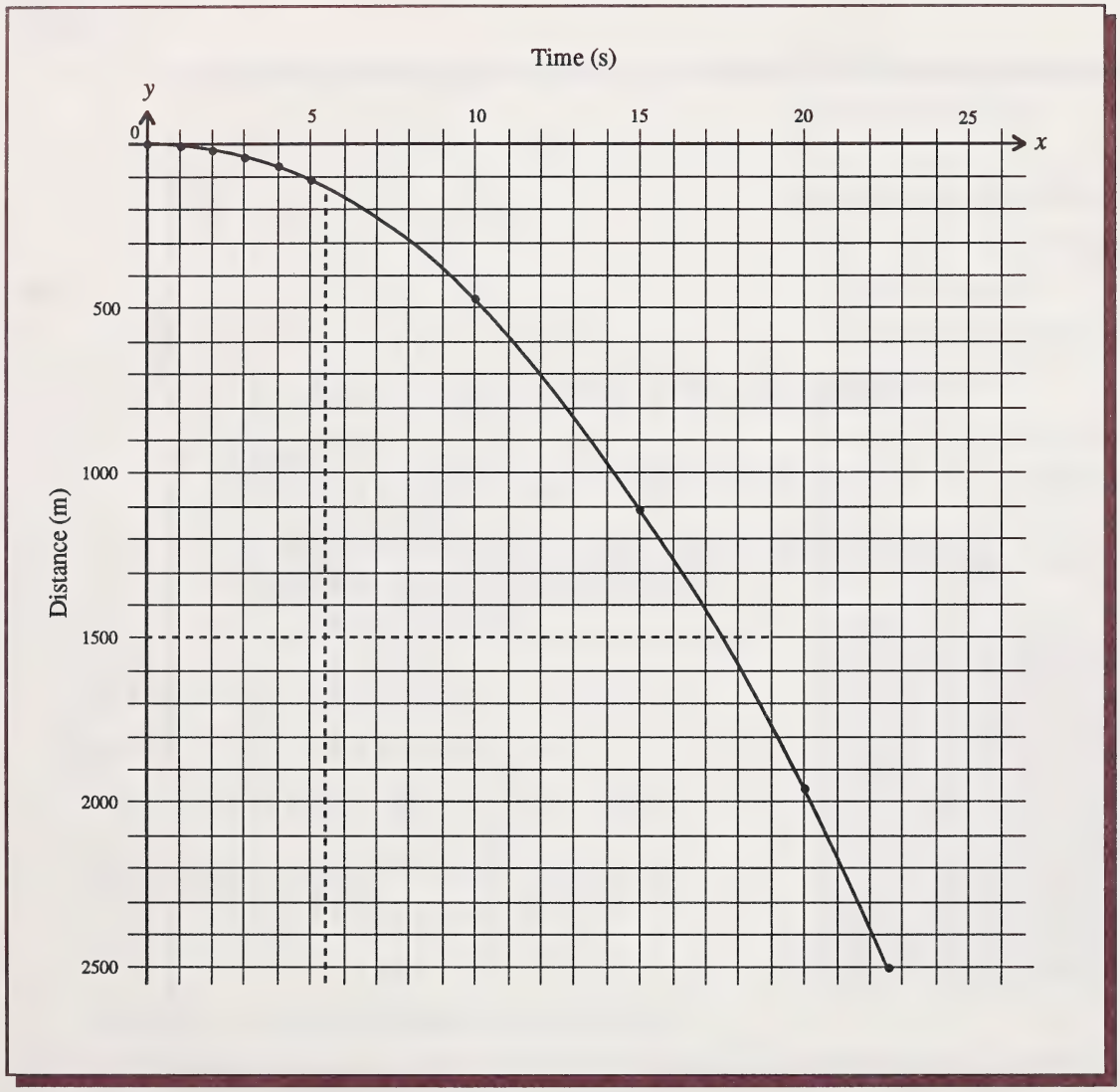
$$t = 1 \text{ s}, \ d = \frac{1}{2}\left(9.8 \text{ m/s}^2\right)(1 \text{ s})^2$$
$$= 4.9 \text{ m}$$
$$t = 2 \text{ s}, \ d = \frac{1}{2}\left(9.8 \text{ m/s}^2\right)(2 \text{ s})^2$$
$$= \left(4.9 \text{ m/s}^2\right)\left(4 \text{ s}^2\right)$$
$$= 19.6 \text{ m}$$
$$t = 3 \text{ s}, \ d = \frac{1}{2}\left(9.8 \text{ m/s}^2\right)(3 \text{ s})^2$$
$$= 44.1 \text{ m}$$

Two ways to represent the distance are as follows:

- Start at 2500 m for $t = 0$ s and increment down to 0 m down the left side.
- Start at 0 m for $t = 0$ s and increment to 2500 m down the left side.

The second way has been used on the following graph because it is easier to see how far the object has fallen.

Plot the points from the table and join the points with a smooth curve.



The scale starts at zero metres and increases downward. Usually the downward direction indicates negative values, but in this case an exception is made for an easier reading of the graph.

- From the graph determine the time required for the object to fall to the ground.

  Solution:

  In order to interpolate the time taken to fall 2500 m, find the point at which the graph intersects the horizontal line where the distance equals 2500 m.

  $t \doteq 22.6$ s

  Check:
  When $t = 22.6$ s, the distance fallen is as follows:

  $$d = \frac{1}{2}\left(9.8 \text{ m/s}^2\right)(22.6 \text{ s})^2$$
  $$= \left(4.9 \text{ m/s}^2\right)\left(510.76 \text{ s}^2\right)$$
  $$= 2502.7 \text{ m}$$

  This is very close to the required distance of 2500 m. Therefore, the time required for the object to reach the ground is about 22.6 s.

- From the graph determine the distance the object had fallen after 5.5 s.

  Solution:

  In order to find the distance the object fell after 5.5 s, draw a vertical line at $t = 5.5$ s and read the distance at the point of intersection with the graph. (See the vertical dotted line on the graph.)

  $d \doteq 150$ m

  If you read the distance as 2350 m, remember to subtract this from 2500 m in order to get the distance the object fell. Therefore, the distance the object fell was about 150 m.

- From the graph determine the time required for the object to fall a distance of 1500 m.

  Solution:

  Draw a horizontal line at $d = 1500$ m. (See the horizontal dotted line.)

  $t \doteq 17.5$ s

  The time required for the object to fall 1500 m is about 17.5 s.

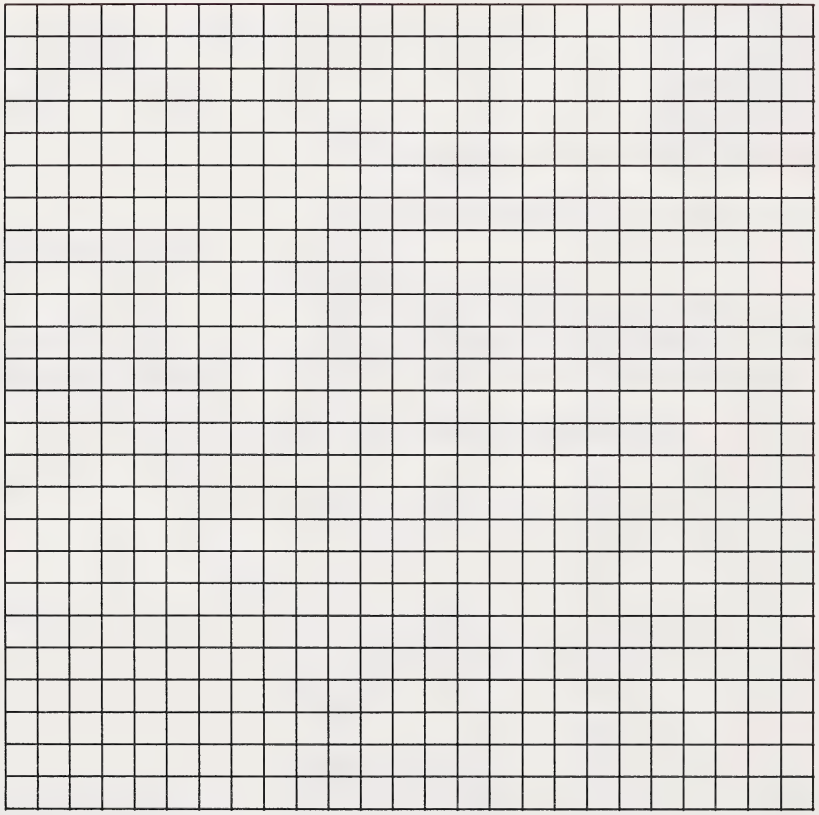
Do any three of the following questions.

1. People's reaction times are related to their ages. A table of hypothetical reaction times for various age groups is given. Draw a graph of reaction time versus age. From the graph determine the following:

   a. What is the reaction time for a person twelve years old?

   b. What is the age of a person whose reaction time is 0.35 s?

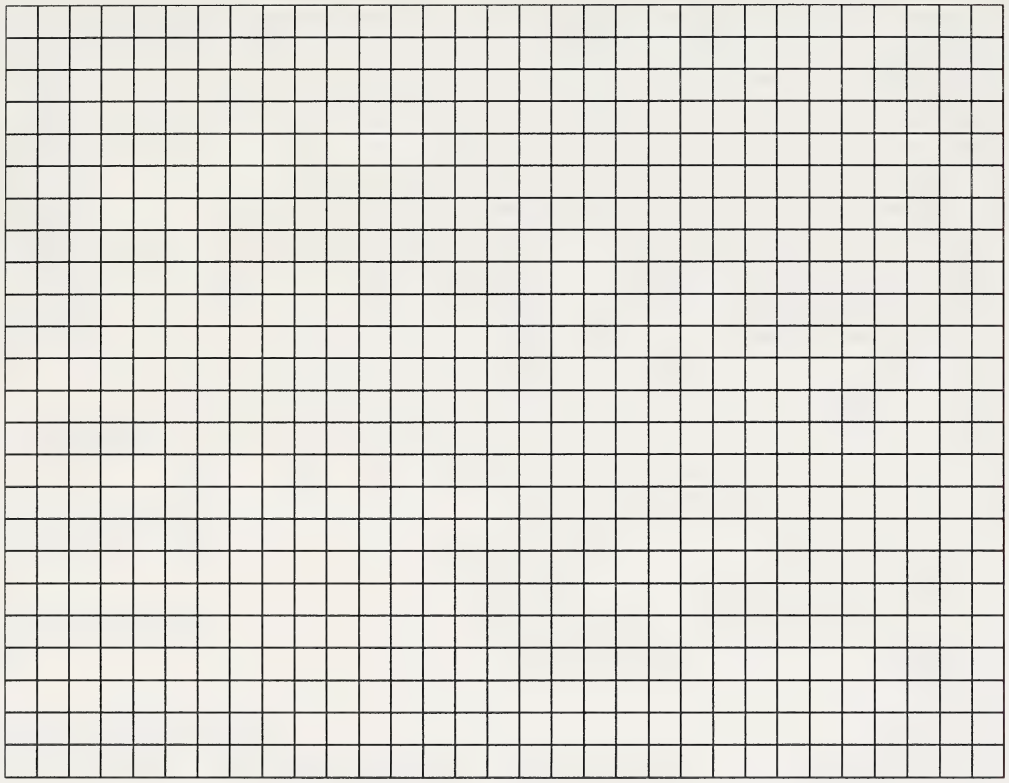
Age (a)	Reaction time (s)
1	2.00
2	1.00
3	0.75
5	0.50
15	0.25
20	0.20
25	0.20
30	0.25
35	0.30
40	0.40
45	0.50
50	0.60
55	0.70
60	0.80
65	0.90
70	1.00
75	1.50
80	2.00



2. Susan earns $7.25 per hour. Draw the graph of earnings versus time worked in hours. Label the graph to 60 h.

   a. According to the graph, how long would Susan have to work in order to earn $348?

   b. If Susan works for 37 h, how much would she earn?



3. A signal flare is fired from the ground upwards. The initial speed of the flare is 100 m/s. Use the formula $h = v_i t + \frac{1}{2} a t^2$ to produce a table of values for time and height. Draw the graph of distance versus time using this table of values.

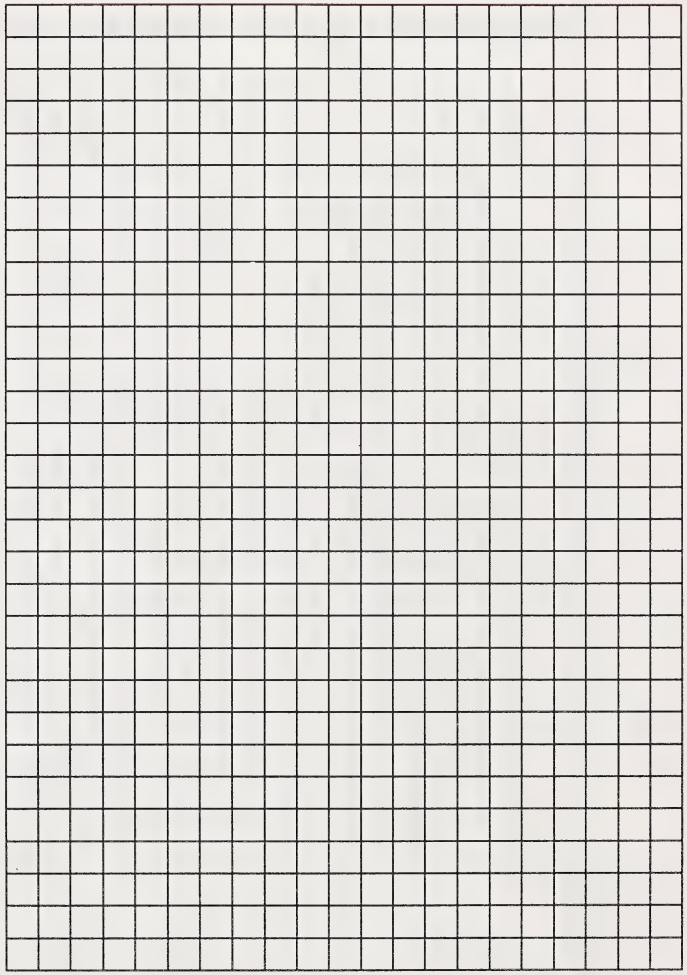
h = height in metres

v_i = initial speed in metres per second

t = time in seconds

a = acceleration due to gravity $\left(-9.8 \text{ m/s}^2\right)$

Time (s)	Height (m)



a. From the graph determine how high the flare will travel.

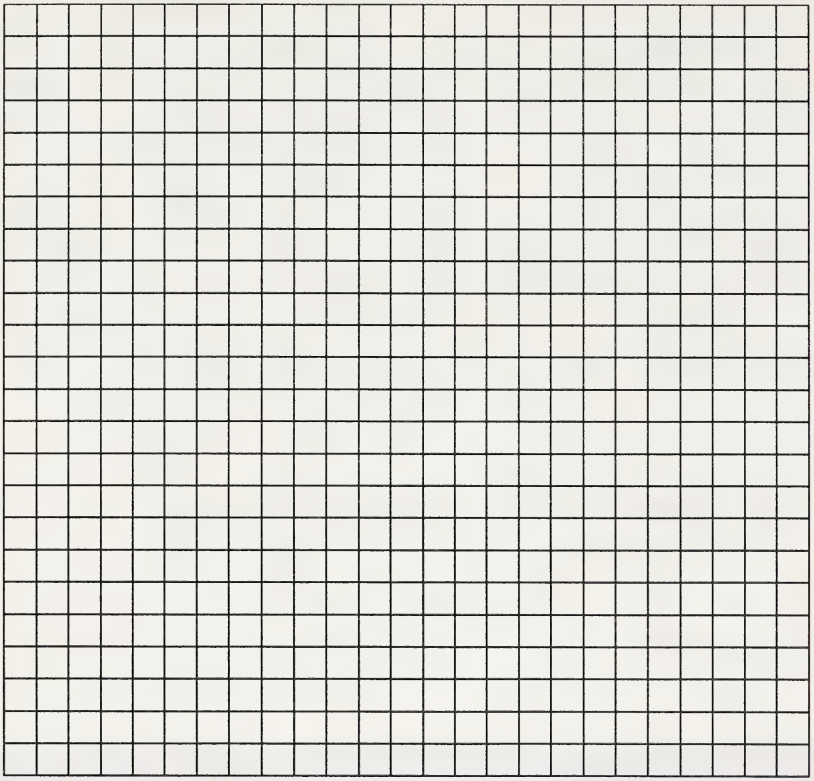
b. From the graph determine the times at which the height will be 250 m.

4. Ocean farming is becoming more widespread. The data in the table relates the mass of a specific fish in kilograms to its age in months. Draw a graph of this relation. Join the points with a smooth curve.

   a. According to the graph, how old will a fish have to be to have a mass of 900 g? (Hint: 0.800 kg = 800 g.)

   b. According to the graph, what is the mass of a fish which is three years old?

Age (mo)	Mass (kg)
0	0.000
1	0.050
2	0.150
3	0.300
4	0.425
5	0.525
6	0.600
12	0.800
24	1.000
40	1.150



For solutions to **Activity 2**, turn to **Appendix A, Topic 1**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

Graphing relations can be done simply by doing the following:

- Make a table of values.
- Plot each point from the table of values.

Example 7

Draw the graph of velocity (v) versus time (t) where $v = (10\ \text{m/s}^2)t$. Let the domain be 0 to 10.

Solution:

The independent variable is time and the dependent variable is velocity. Calculate v using increments of one second and make a table of values.

The domain is all of the values that can be used for the independent variable.

The range is all of the values that can be used for the dependent variable.

$t = 0\ \text{s},\ v = (10\ \text{m/s}^2)(0\ \text{s}) = 0\ \text{m/s}$

$t = 1\ \text{s},\ v = (10\ \text{m/s}^2)(1\ \text{s}) = 10\ \text{m/s}$

$t = 2\ \text{s},\ v = (10\ \text{m/s}^2)(2\ \text{s}) = 20\ \text{m/s}$

$t = 3\ \text{s},\ v = (10\ \text{m/s}^2)(3\ \text{s}) = 30\ \text{m/s}$

$t = 4\ \text{s},\ v = (10\ \text{m/s}^2)(4\ \text{s}) = 40\ \text{m/s}$

$t = 5\ \text{s},\ v = (10\ \text{m/s}^2)(5\ \text{s}) = 50\ \text{m/s}$

$t = 6\ \text{s},\ v = (10\ \text{m/s}^2)(6\ \text{s}) = 60\ \text{m/s}$

$t = 7\ \text{s},\ v = (10\ \text{m/s}^2)(7\ \text{s}) = 70\ \text{m/s}$

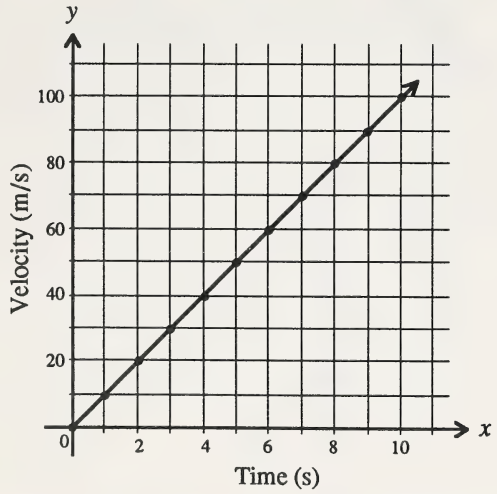
$t = 8\ \text{s},\ v = (10\ \text{m/s}^2)(8\ \text{s}) = 80\ \text{m/s}$

$t = 9\ \text{s},\ v = (10\ \text{m/s}^2)(9\ \text{s}) = 90\ \text{m/s}$

$t = 10\ \text{s},\ v = (10\ \text{m/s}^2)(10\ \text{s}) = 100\ \text{m/s}$

Time (s)	Velocity (m/s)
0	0
1	10
2	20
3	30
4	40
5	50
6	60
7	70
8	80
9	90
10	100

Plot the table of values using each time-velocity pair as a point on the graph.



Once the graph is plotted, you can interpolate and extrapolate. You can draw a vertical line from a required time so that it intersects the graph. Then draw a horizontal line from this point of intersection to the velocity axis. If you are given a velocity, you can draw a horizontal line to intersect the graph and then a vertical line from this point to the time axis.

Example 8

From the graph in Example 7, determine the following:

- the velocity when the time = 4.5 s

  Solution:

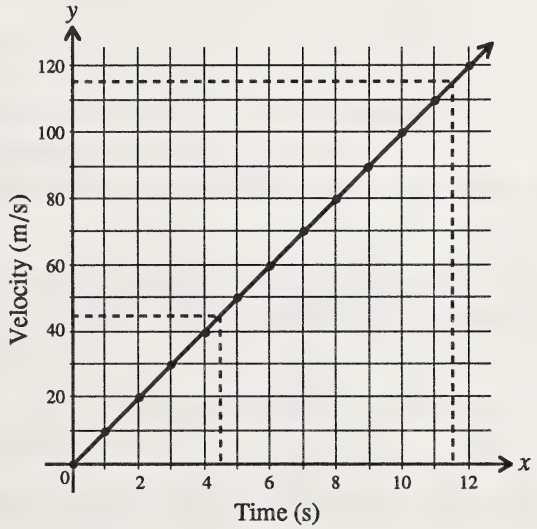
  Draw a vertical line from $t = 4.5$ s to intersect the graph. From the point of intersection draw a horizontal line to the velocity axis. The velocity is 45 m/s.

- the time when the velocity = 115 m / s

  Solution:

  Draw a horizontal line from $v = 115$ m / s to intersect the graph. From the point of intersection draw a vertical line to the time axis. The time is 11.5 s.

x	y
1	10
2	20
3	30
4	40
5	50
6	60
7	70
8	80
9	90
10	100
11	110
12	120



Try the following questions. Graph paper is provided in **Appendix B.**

1. Draw the graph of acceleration (a) versus time (t) where

   $a = \frac{50 \text{ m/s}}{t}$. Let the domain be zero to ten seconds.

2. From the graph determine the following:

   a. the acceleration when $t = 1.5$ s

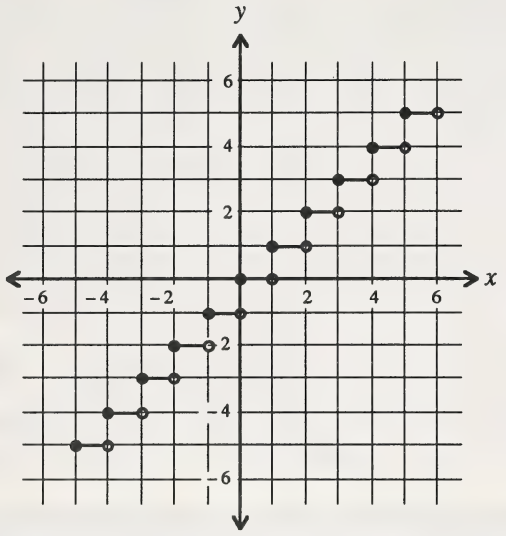
   b. the time when acceleration is 20 m/s^2

For solutions to **Extra Help**, turn to **Appendix A, Topic 1.**

Extensions

You have drawn the graphs of relations that have only distinct point values such as the relation between the number of chocolate bars and their total cost. You cannot buy fractions of chocolate bars. Also, you have drawn graphs of relations that contain the entire range of values and these graphs resulted in a curve which joined the points.

There are relations that require yet another graphing technique. The step function is one of these. Below is the graph of $y = [x]$.



∘ means the point is not included in the graph.
• means the point is included in the graph.

Notice that all possible x-values are used in the graph. In other words, $x \in R$. Only integer values of y are displayed on the graph. When x is greater than or equal to 0, but less than 1, $y = 0$. In mathematical symbols, when $0 \leq x < 1$, $y = 0$. You can write the relationship for all values of x and y in this way. In other words, the relationship $y = [x]$ means the greatest integer less than or equal to x. To show that $x \in R$, take $5\frac{1}{2}$ and $-1\frac{1}{8}$ on the x-axis. Move down or up from these points to see if contact is made with the graph. Contact would be made no matter which x-value was chosen. Now take $3\frac{3}{4}$ and $-4\frac{1}{2}$ on the y-axis. Move to the right or to the left from these points to see if contact is made with the graph. No contact for such values is possible. The only points of contact are the integers such as 1, 4, – 5, – 4, 5, and so on.

Try the following question.

The taxi fare in Example 2 of this topic could have been drawn as a step function. As an exercise, draw the graph of Example 2 using the new graphing information presented here. Graph paper is provided in **Appendix B.**

For solutions to **Extensions**, turn to **Appendix A, Topic 1.**

Topic 2 Functions

Introduction

A **function** is a special kind of relation. All functions are relations, but not all relations are functions.

A simple analogy that can illustrate this is trees.

A spruce tree is a special kind of tree. All spruce trees are trees, but not all trees are spruce trees.

What Lies Ahead

Throughout the topic you will learn to

1. represent simple mathematical situations such as direct and inverse variations with tables of values, identify the dependent and independent variables, and express the domain and the range noting any restrictions

2. write functions using functional notation and evaluate them by substitution for the domain

3. graphically and algebraically illustrate and define different kinds of functions, in particular, polynomial (linear, quadratic, and cubic), reciprocal, absolute-value, square-root, and exponential functions

4. solve problems involving functions algebraically and by drawing, interpolating, extrapolating, interpreting, and analyzing graphs of functions

Now that you know what to expect, turn the page to begin your study of functions.

Exploring Topic 2

Activity 1

Represent simple mathematical situations such as direct and inverse variations with tables of values, identify the dependent and independent variables, and express the domain and the range noting any restrictions.

There are special relations called functions. A **function** is a relation in which no two ordered pairs have the same first value. This definition is used to determine whether or not a relation is a function.

Example 1

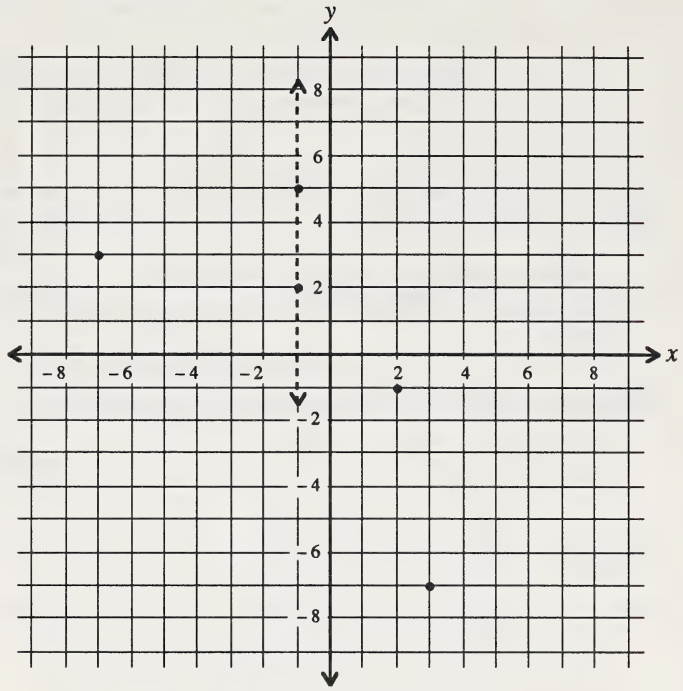
Is this relation a function?

$(-1, 2),\ (3, -7),\ (-7, 3),\ (2, -1),\ (-1, 5)$

Solution:

The relation is not a function since there are two ordered pairs in the list which have the same first coordinate.

Look at the graph of this relation.

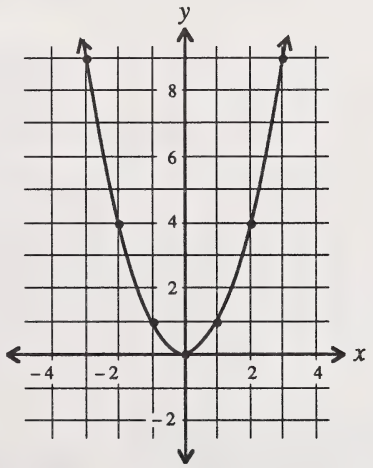


Notice that a vertical line drawn through the point $(-1, 2)$ also passes through the point $(-1, 5)$. This gives rise to a test to determine whether a relation is also a function.

Vertical-line test: If any vertical straight line passes through more than one point on the graph, the relation is not a function.

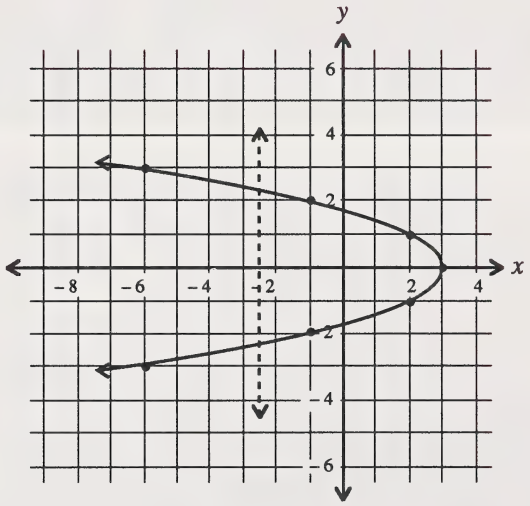
Example 2

Determine whether or not each of these graphs of a relation is a function.



Solution:

This relation is a function since all vertical lines through the graph pass through only one point on the graph.



Solution:

This relation is not a function since any vertical line would pass through the graph at two points.

Equation test: If you can find any one value of x which produces two values of y, the relation is not a function.

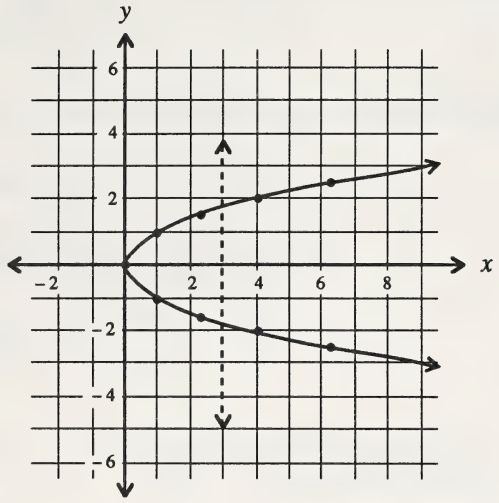
Example 3

Is $y^2 = x$ a function?

Solution:

No, it is not a function since $y = \pm\sqrt{x}$; each value of x produces two values of y.

The graph of this relation shows that it is not a function according to the vertical-line test.

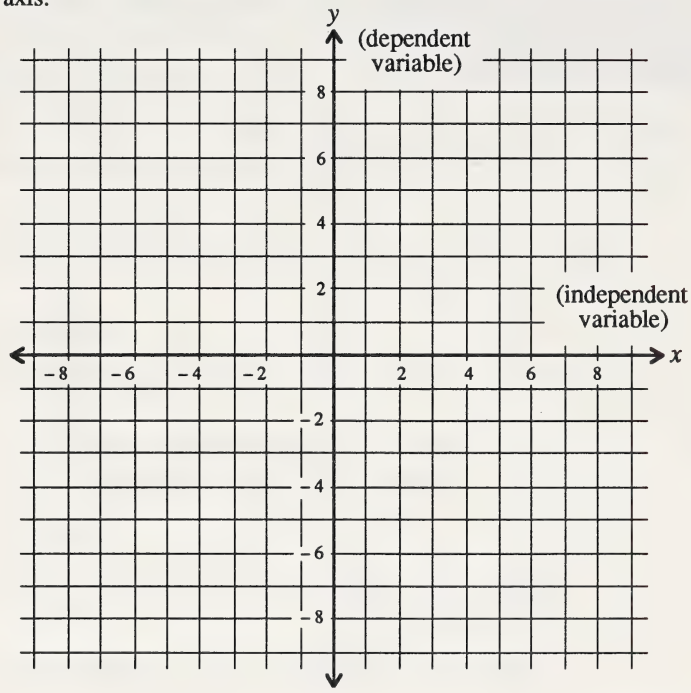


Notice that x cannot have negative values since the square root of a negative number is not a real number.

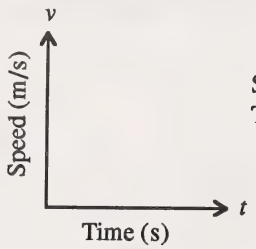
Functions written as equations will contain two variables, x and y. These variables are used to maintain a consistency of presentation and are not intended to be used exclusively.

The x-variable is called the **independent variable.**
The y-variable is called the **dependent variable.**

This means that the value of y depends on the value of x.
The independent variable (x) is always graphed on the horizontal axis. The dependent variable (y) is always graphed on the vertical axis.



If a function involves variables other than x and y, the independent variable must be plotted on the horizontal axis and the dependent variable must be plotted on the vertical axis.



Speed (v) is the dependent variable.
Time (t) is the independent variable.

The **domain** is the set of all possible x-values.
The **range** is the set of all possible y-values.

Example 4

Determine the domain and the range of the function $y = 3x + 2$.

Solution:

Any value from the set of real numbers can be substituted for x; therefore, the domain is $x \in R$.
($x \in R$ is used because x is a real number.)

The range is $y \in R$ because for any real number replacement for y, you can find a replacement for x in the equation that will produce that y-value.

Example 5

For the equation $y = x^2 - 3$, determine the following.

- the dependent variable and the independent variable

  Solution:

  The dependent variable is the y-variable.
  The independent variable is the x-variable.

- the domain and the range

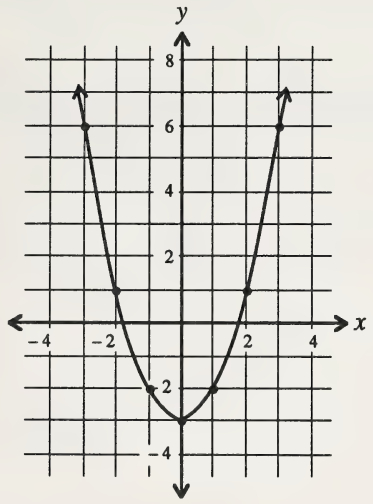
  Solution:

  The domain of the function $y = x^2 - 3$ is the set of real numbers, $x \in R$.

  The range is the set of real numbers greater than or equal to -3. In symbols the range is defined by $y \geq -3$, $y \in R$.

Why does the range in Example 5 have the restriction that y must be greater than or equal to -3? Look at the graph of the function. First make a table of values, then plot the graph.

x	y
-3	6
-2	1
-1	-2
0	-3
1	-2
2	1
3	6



Notice that the y-values never go below -3.

Do all of questions 1 to 5 and any two of questions 6 to 10.

1. Which of the following groups of ordered pairs represent functions?

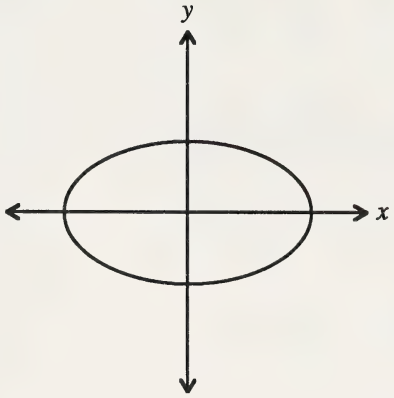
   a. $\{(0,-7), (-3,5), (4,1), (-2,5), (0,0)\}$

   b. $\left\{\left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{3},\frac{1}{3}\right), \left(\frac{1}{4},\frac{1}{4}\right), (2,2)\right\}$

   c. $\left\{\left(\sqrt{6},2\right), \left(\frac{1}{3},5\right), \left(-\sqrt{6},2\right), \left(-\frac{1}{3},5\right)\right\}$

2. Which of the following graphs represent functions?

   a.



Sample calculations are as follows:

If $x=-3$, $y=(-3)^2-3$

$y=9-3$

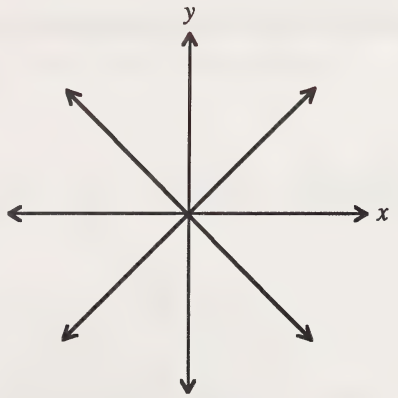
$y=6$

If $x=-2$, $y=(-2)^2-3$

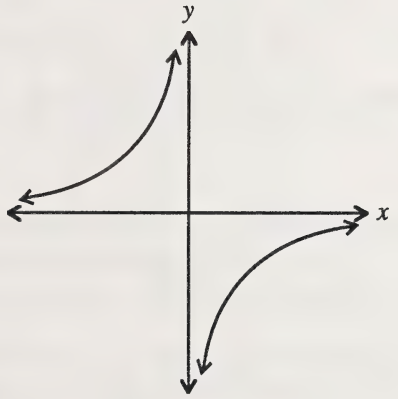
$y=4-3$

$y=1$

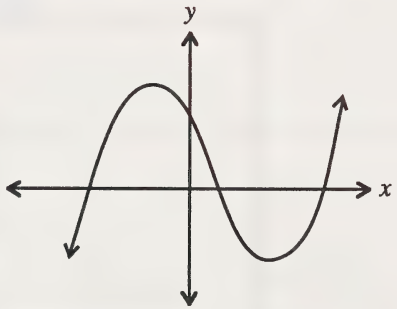
b.



c.



d.



3. Which of the following equations represent functions?

a. $y=-2x+1$

b. $y^2-x=0$

c. $3y+x-5=0$

d. $x^2+y^2=25$

e. $d=\frac{1}{2}at^2$, where a is a constant $\left(-9.8\ \text{m/s}^2\right)$

f. $y=x^2$

g. $y=\frac{1}{x^2}$

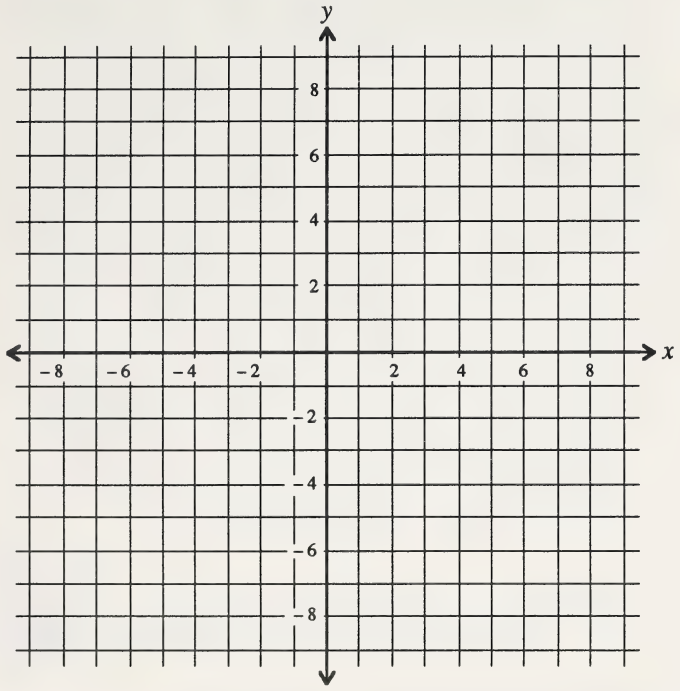
h. $y=\sqrt{x}+1$

4. For each of the question parts in question 3, state the dependent variable and the independent variable.

5. For each of the question parts in question 3, state the domain and the range.

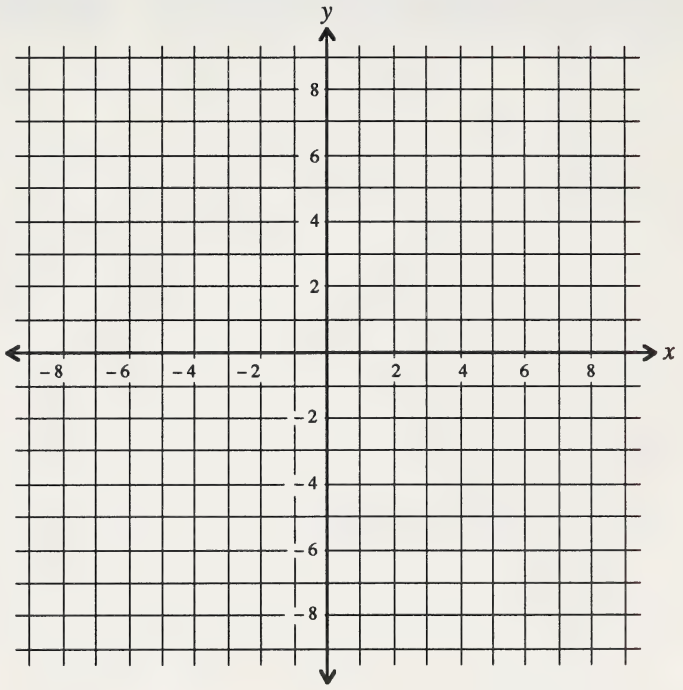
6. Draw the graph of $2y^2 - 4x = 4$.

   Is this a function? State the domain and the range.



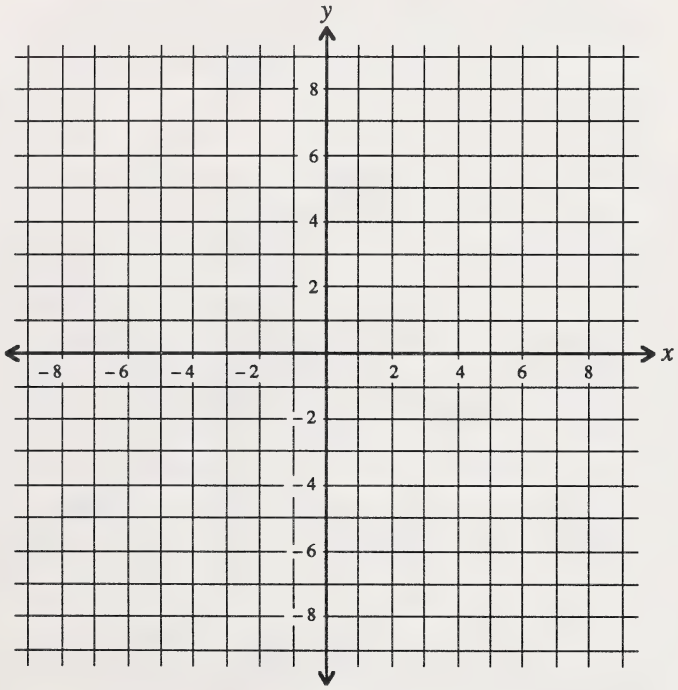
7. Draw the graph of $y = -2x^2 + 1$.

   Is this a function? State the domain and the range.



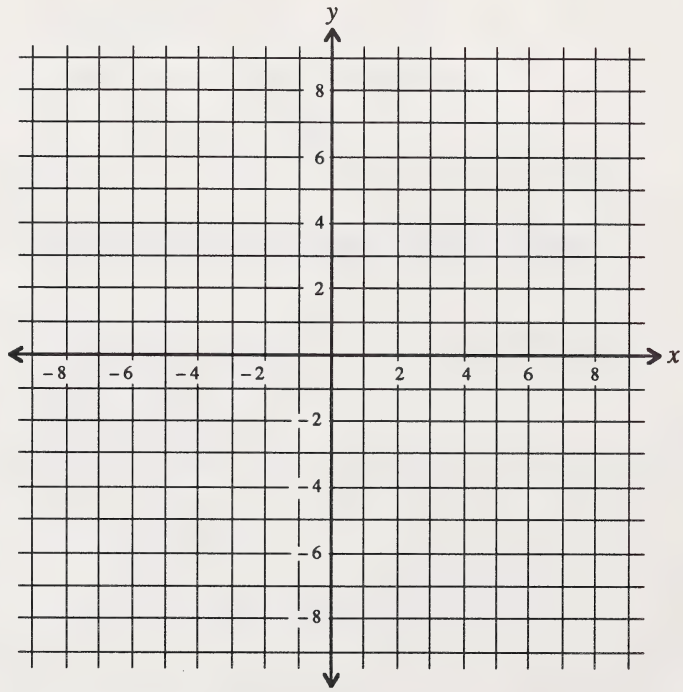
8. Draw the graph of $y = \frac{6}{2x+1}$.

Is this a function? State the domain and the range.



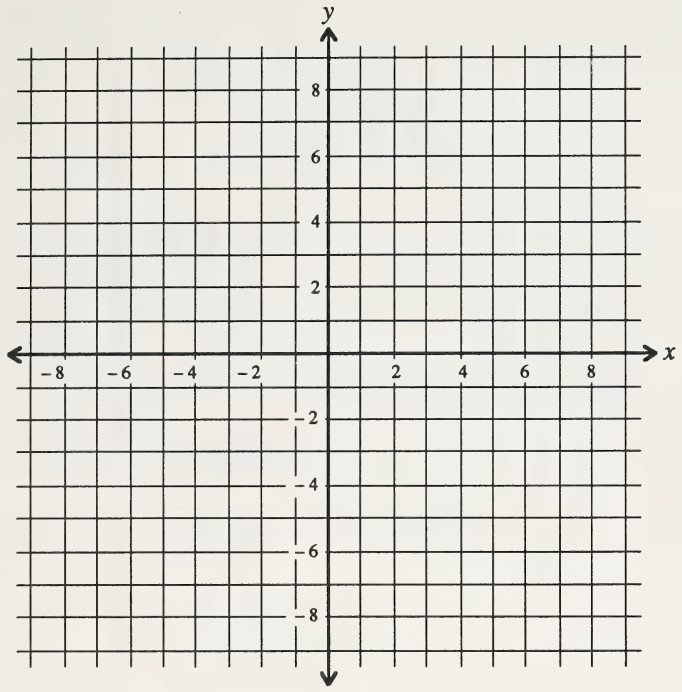
9. Draw the graph of $3xy = 9$.

Is this a function? State the domain and the range.



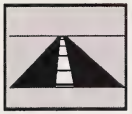
10. Draw the graph of $\frac{1}{x}+2=y$.

Is this a function? State the domain and the range.



For solutions to **Activity 1**, turn to **Appendix A, Topic 2.**

Activity 2



Write functions using functional notation and evaluate them by substitution for the domain.

Often functions are evaluated for certain values of x.

Example 6

Evaluate $y=2x-3$ when $x=5$.

Solution:

$y=2(5)-3$
$y=10-3$
$y=7$

This evaluation process occurs so often that a simpler notation was developed. The equation $y=2x-3$ is written as $f(x)=2x-3$. The variable y is replaced by the symbol $f(x)$ or $g(x)$ or any other letter. The notations $f(x)$ and $g(x)$ are most commonly used. How does this make it simpler? Evaluate $y=2x-3$ when $x=5$.

$f(x)=2x-3$
$f(5)=2(5)-3$
$=7$

Example 7

Given that $g(x)=x^2-7$, find $g(4)$.

Solution:

$$g(4)=4^2-7$$
$$g(4)=16-7$$
$$g(4)=9$$

The symbol $g(x)$ is read ***g* of *x***.

Example 8

If $P(x)=2x^2+3x+1$, evaluate the following.

- $P(-1)$

  Solution:

$$P(-1)=2(-1)^2+3(-1)+1$$
$$=2(1)+3(-1)+1$$
$$=2-3+1$$
$$=0$$

- $P(a)$

  Solution:

$$P(a)=2(a)^2+3(a)+1$$
$$=2a^2+3a+1$$

- $P(5)$

  Solution:

$$P(5)=2(5)^2+3(5)+1$$
$$=2(25)+3(5)+1$$
$$=50+15+1$$
$$=66$$

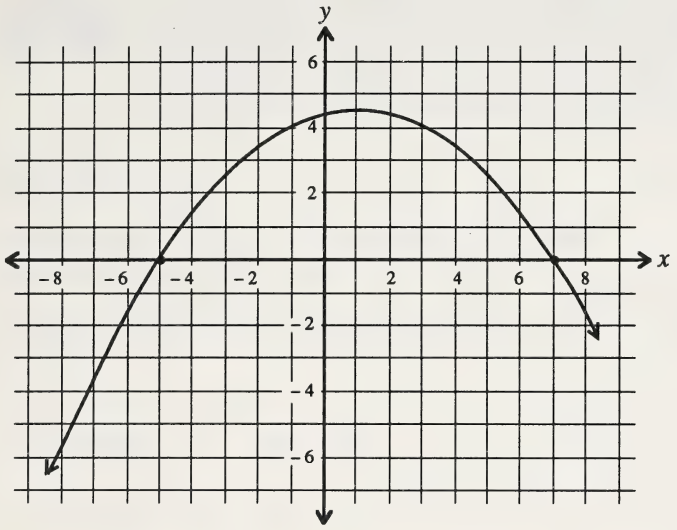
- $P(z+1)$

  Solution:

$$P(z+1)=2(z+1)^2+3(z+1)+1$$
$$=2\left(z^2+2z+1\right)+3(z+1)+1$$
$$=2z^2+4z+2+3z+3+1$$
$$=2z^2+7z+6$$

Example 9

Given the graph of $y = g(x)$, determine $g(-5)$ and $g(3)$.



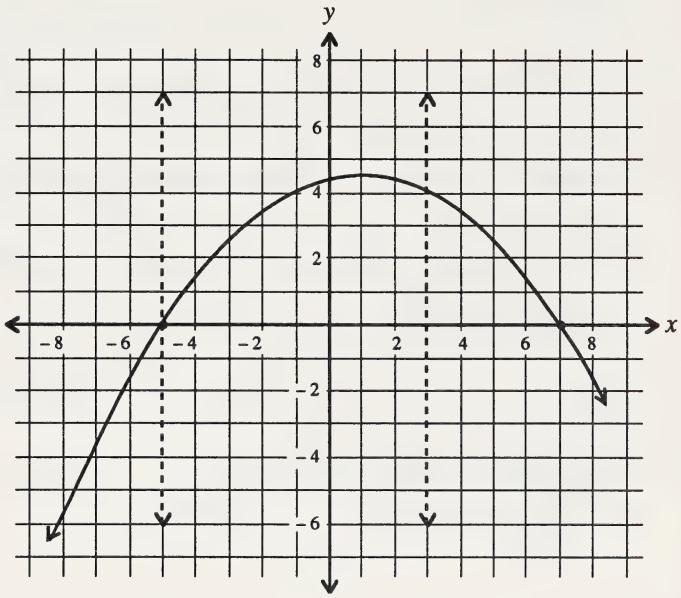
Solution:

To find $g(-5)$, draw a vertical line $x = -5$ and estimate the intersection of the line and the graph.

$g(-5) = 0$

To find $g(3)$, draw a vertical line $x = 3$ and estimate the intersection of this line with the graph.

$g(3) = 4$



Do any two of the following questions.

1. Given $f(x)=\dfrac{2x}{x-6}$, evaluate each of the following.

   a. $f(-8)$ b. $f(0)$

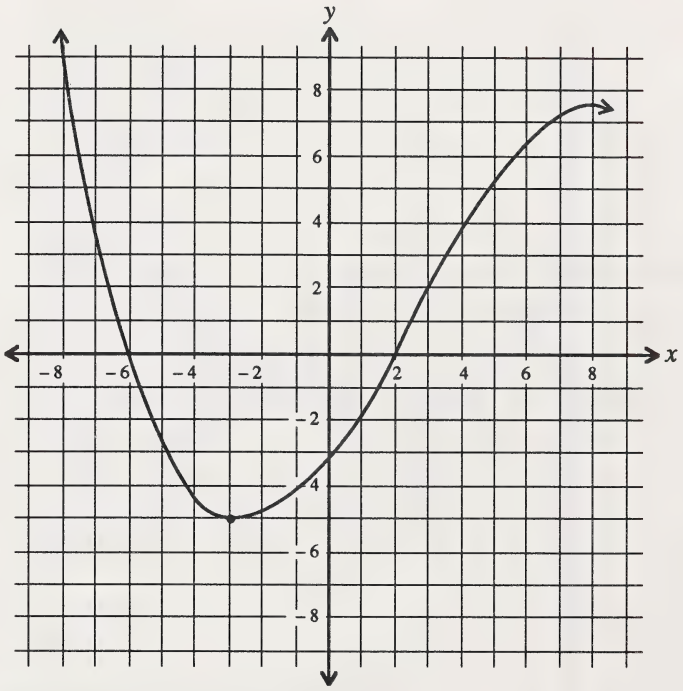
   c. $f\left(x^2\right)$ d. $f(-1)+f(3)$

2. Given $g(x)=2x^3-x^2+5x-1$, evaluate each of the following.

   a. $g(-1)$ b. $g(2)$

   c. $g(0)$ d. $g(a)$

3. Given the graph of the function $h(x)$, evaluate $h(-3)$.



For solutions to **Activity 2**, turn to **Appendix A, Topic 2.**

Activity 3

Graphically and algebraically illustrate and define different kinds of functions, in particular, polynomial (linear, quadratic, and cubic), reciprocal, absolute-value, square-root, and exponential functions.

Polynomial Functions

A polynomial function is a function in which the expression involving the independent variable is a polynomial.

The following are examples of polynomial functions.

- $y = 2x + 1$
- $f(x) = -3x^2 + 2$
- $y = x^4 - x^3 + x^2 - x - 1$
- $g(x) = 3x^5 - 5$

The following are examples of functions which are not polynomial functions.

- $y = x^{\frac{1}{2}}$ (A polynomial may only have terms in which the variable is raised to whole number exponents.)
- $y = \frac{1}{x}$ (This is the same as $y = x^{-1}$; the exponent is not a whole number.)
- $f(x) = 2^x$ (In a polynomial the variable cannot occur as an exponent; it must be the base.)

In this section you will study the following polynomial functions.

- linear
- quadratic
- cubic

Linear Functions

All **linear functions** have an equation of the general form $y = mx + b$, **where** $m \neq 0$.

They are all first order, that is, the exponent of the independent variable is always one.

The graph of a linear function is always a straight line.

The following are examples of equations of linear functions.

- $y = -x + 3$
- $f(x) = 5x$ (Note in this example the value of b is 0, according to the general form $y = mx + b$.)
- $y = \frac{-x}{7} - \frac{2}{3}$

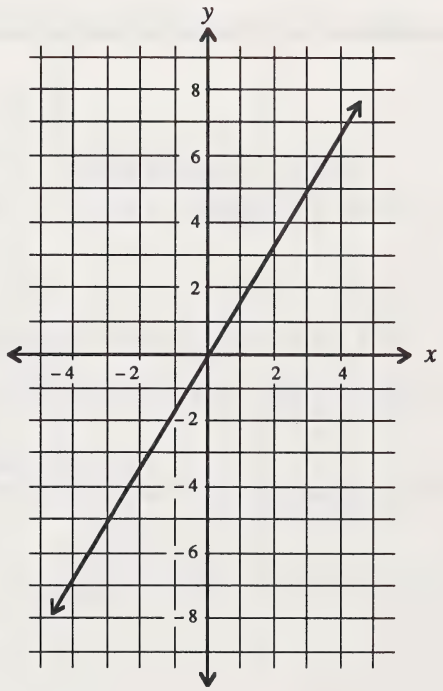
- $2x - y = 3$ (This can be written as $y = 2x - 3$.)

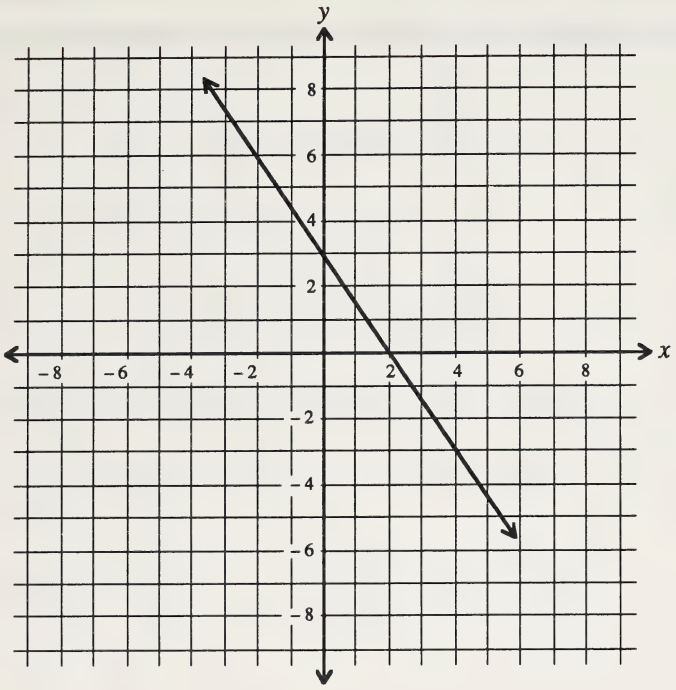
The following are examples of equations that are not linear functions.

- $y = x^2 + 3$ (The degree must be one.)
- $y = \frac{1}{x} + 2$ (This is not even a polynomial function.)
- $f(x) = 3x^3 - 5$ (The degree must be one.)
- $y = |x - 1|$ (This is an absolute-value function.)

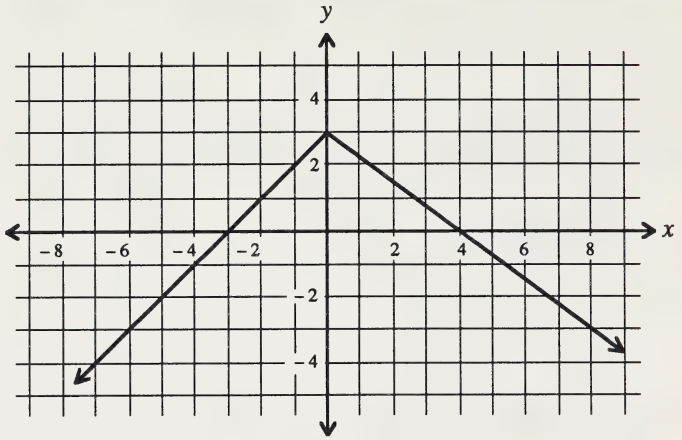
Remember that graphs of linear functions are straight lines.

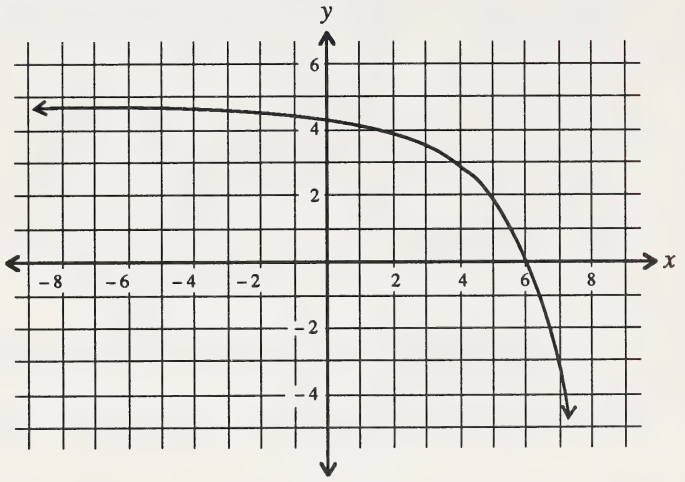
The following are examples of graphs of linear functions.





The following are examples of graphs that are not linear functions.





Quadratic Functions

All **quadratic functions** have the characteristic equation $y = ax^2 + bx + c$, **where** $a \neq 0$.

There must be a term in which the independent variable has an exponent of two. That is why the statement $a \neq 0$ is there.

The following are examples of quadratic functions.

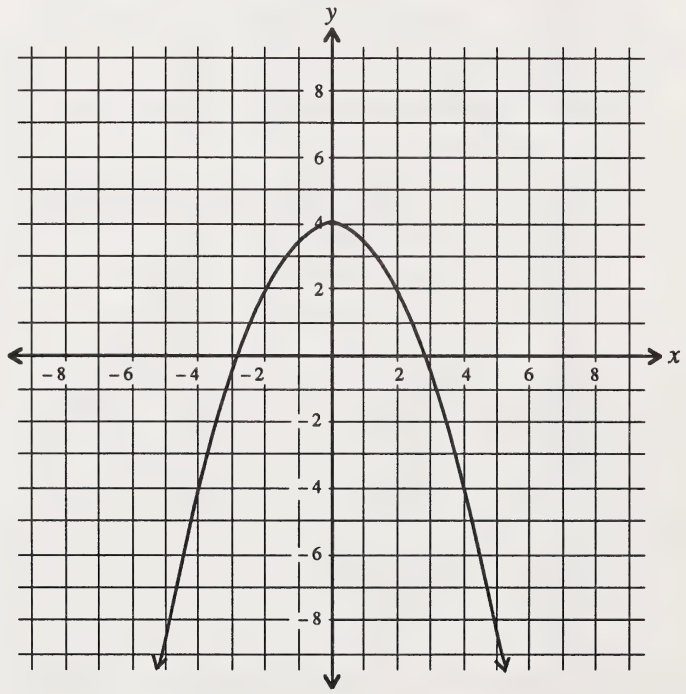
- $y = 5x^2 - 2x + 3$
- $f(x) = -3x^2 + 1$
- $g(x) = x^2$
- $y = \frac{3x^2}{-7} + \frac{4x}{5} - \frac{1}{4}$

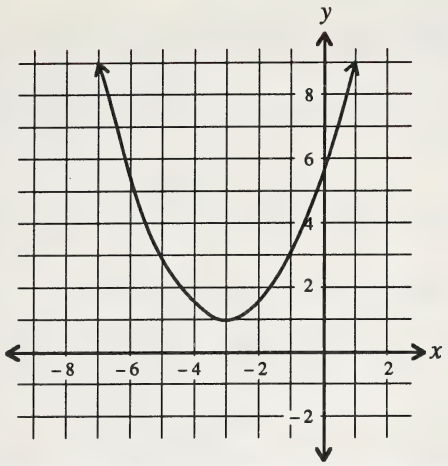
The following are examples of equations which are not quadratic functions.

- $y = x + 3$ (There is no term of degree two.)
- $f(x) = \frac{1}{x^2}$ $\left(\text{This is actually } f(x) = x^{-2}.\right)$
- $y = 5x^3 - 7x^2$ (This is greater than degree two.)
- $g(x) = x^2 + x^{\frac{1}{2}}$ (One term has an exponent other than 2, 1, and 0.)

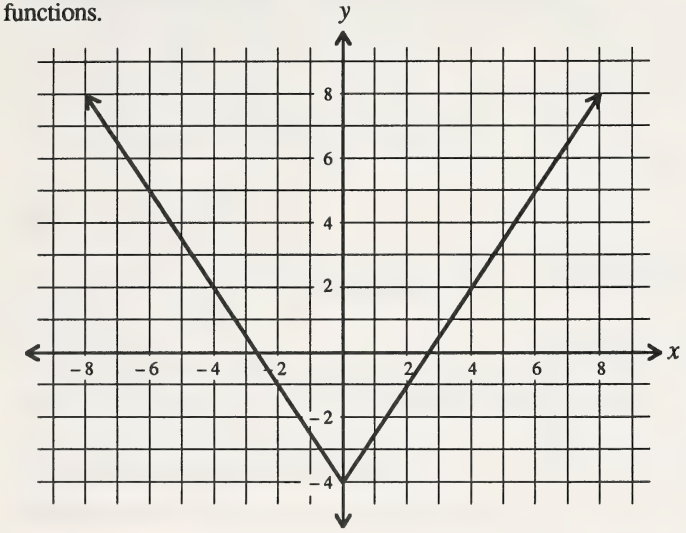
Graphs of quadratics have a characteristic shape called a **parabola**.

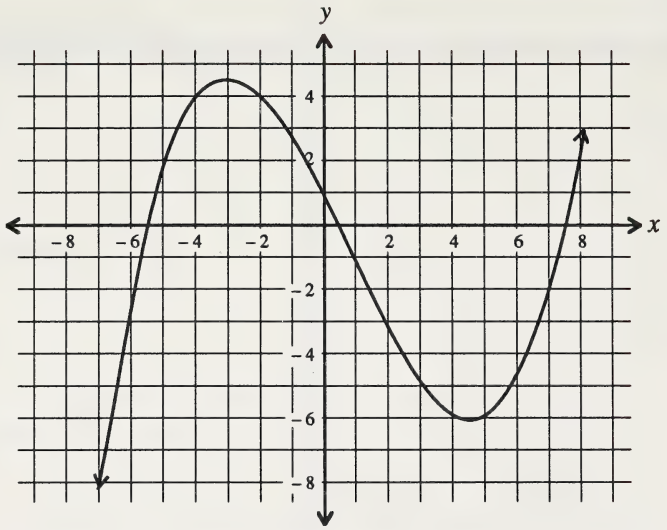
The following are examples of graphs of quadratic functions.



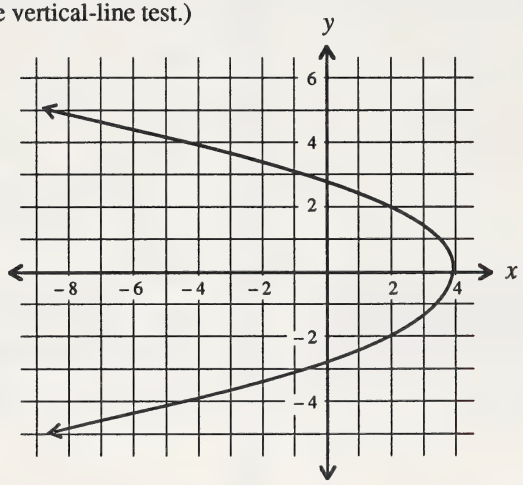


The following are examples of graphs that are not quadratic functions.





The following graph has a parabolic shape, but it is not a function. (Use the vertical-line test.)



Cubic Functions

Cubic functions have the characteristic equation
$y = ax^3 + bx^2 + cx + d$, **where** $a \neq 0$.

There must be a term in which the independent variable has an exponent of three. The degree of the other terms can only be 0, 1, and 2.

The following are examples of cubic functions.

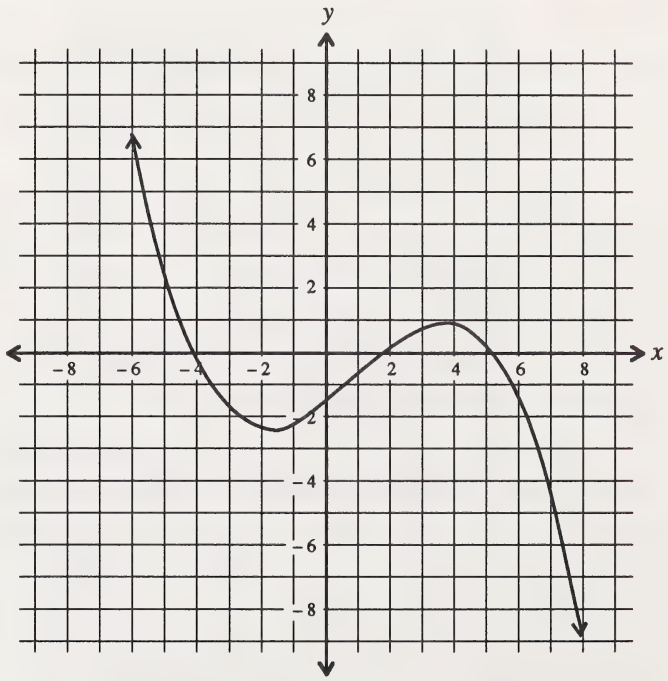
- $y = 2x^3 - x^2 + 3x + 2$
- $y = -3x^3 + x - 2$
- $f(x) = x^3$

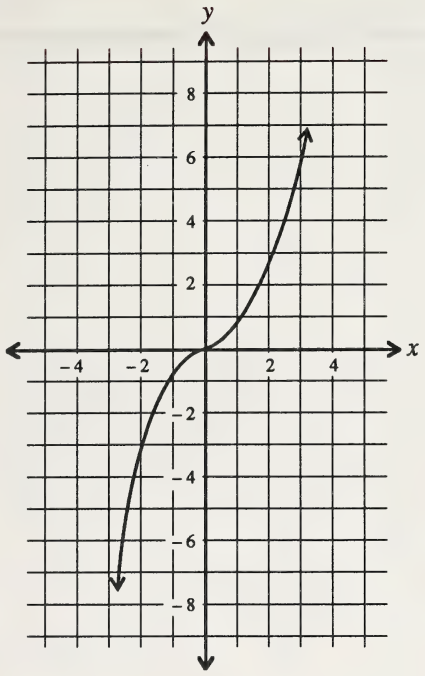
The following are examples that are not cubic functions.

- $y = x^4 + x^3$ (The degree is greater than three.)
- $f(x) = \dfrac{1}{x^3}$ (This can be written as $f(x) = x^{-3}$ and this is not a polynomial with a degree of three.)
- $g(x) = x^3 + x^{\frac{1}{2}}$ (Exponents other than 0, 1, 2, and 3 are not allowed.)

Graphs of cubic equations have a characteristic shape. They look like an S or Z that is on its side.

The following are examples of graphs of cubic functions.





Reciprocal Functions

Reciprocal functions have the following characteristic equation:

$$y = \frac{a}{x}, \text{ where } x \neq 0$$

Any function which has the independent variable in the denominator is a reciprocal function.

The following are examples of reciprocal functions.

- $y = \frac{-5}{x}, x \neq 0$
- $f(x) = \frac{3}{x-2}, x \neq 2 (\text{since } 2-2=0)$
- $g(x) = \frac{1}{x^2}, x \neq 0$

The following are examples that are not reciprocal functions.

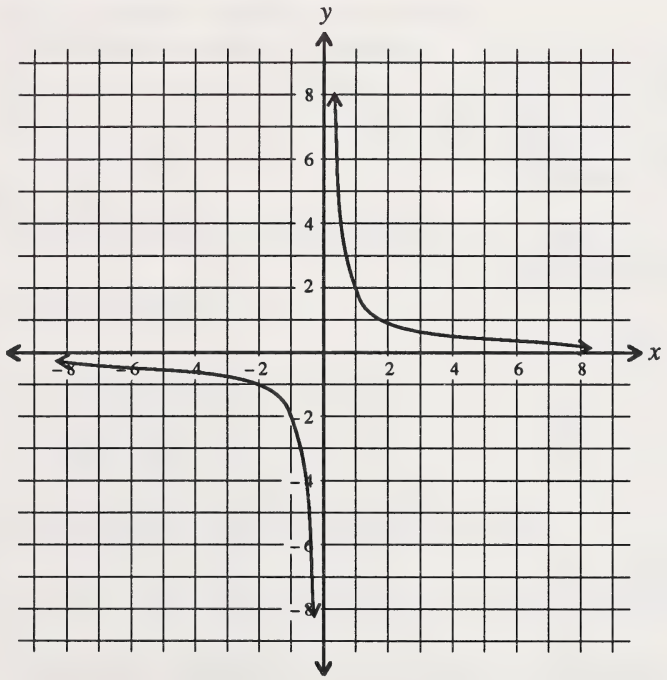
- $y = 2x$
- $f(x) = -x^2 + 3$
- $g(x) = \frac{x^5}{3}$

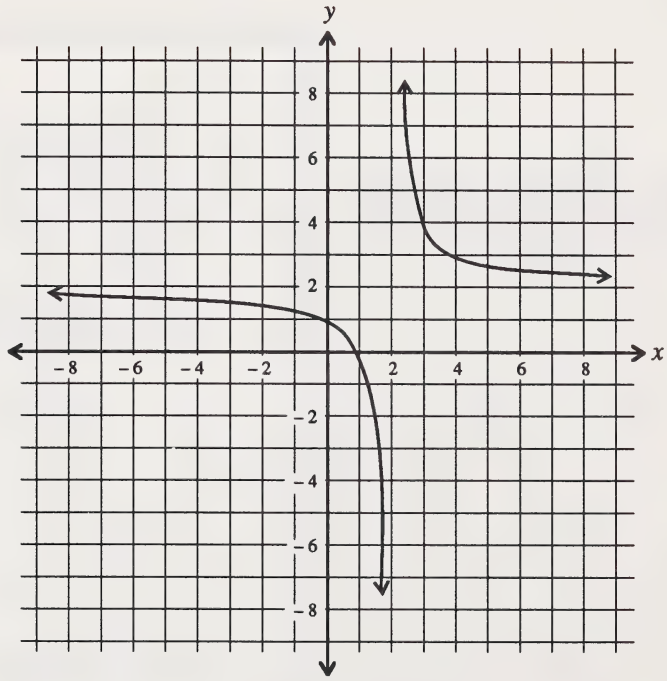
In the previous examples the independent variable is not in the denominator.

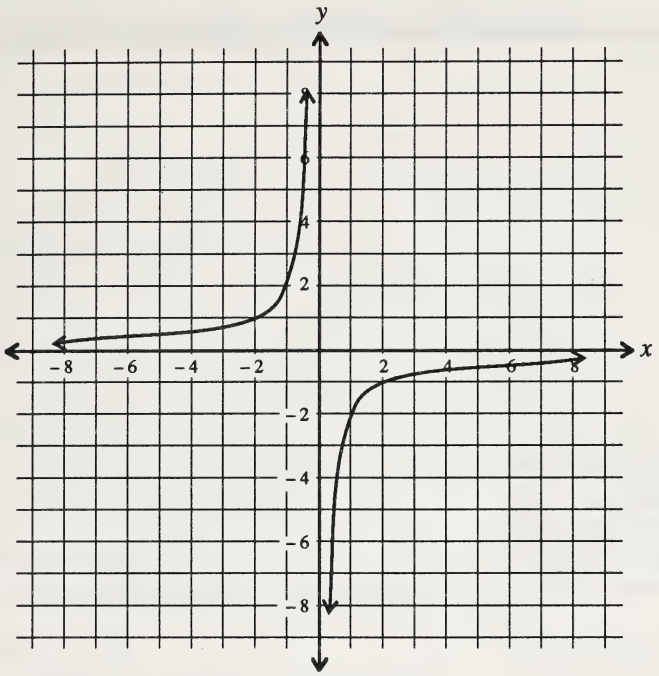
The graphs of reciprocal functions have a characteristic shape called a **hyperbola**. There are always two separate curves separated by the vertical line determined by the value of x which makes the denominator equal to zero.

The reciprocal of x is $\frac{1}{x}$.

The following are examples of the graphs of reciprocal functions.



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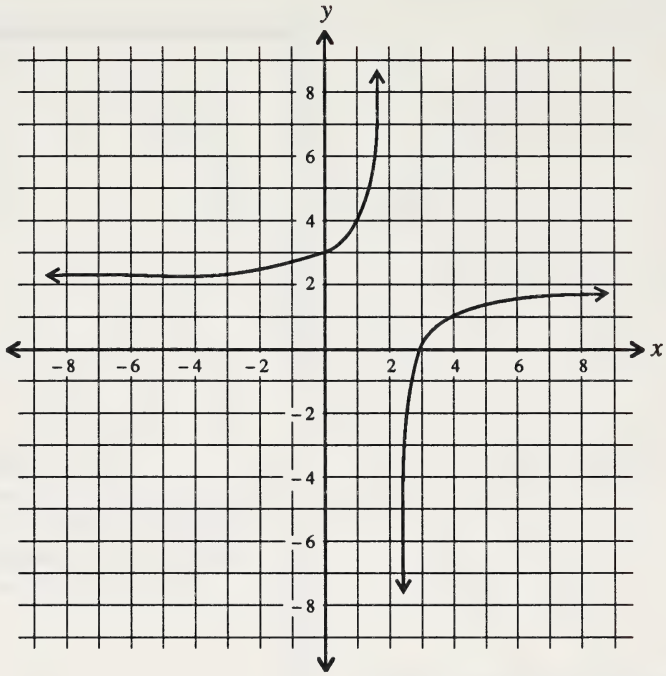
y
x
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−2
−4
−6
−8

Absolute-Value Functions

The **absolute value** of a real number is the value of the number without regard for the sign. The number 5 is the absolute value of $+5$ and -5. Absolute value is often referred to as magnitude. Absolute value refers to the distance from the zero point on a number line.

The symbol for absolute value is $|\ |$.

For example, $|-2|$ means the absolute value of -2.

The following are examples of solutions for absolute value.

- $|-2|=2$
- $|+5|=5$
- $|+2x|=2x$, if $x \geq 0$
- $\dfrac{|-7|}{2x}=\dfrac{7}{2x}$, $x \neq 0$

Absolute-value functions have characteristic equations in the general form $y=|x|$.

The following are examples of absolute-value functions.

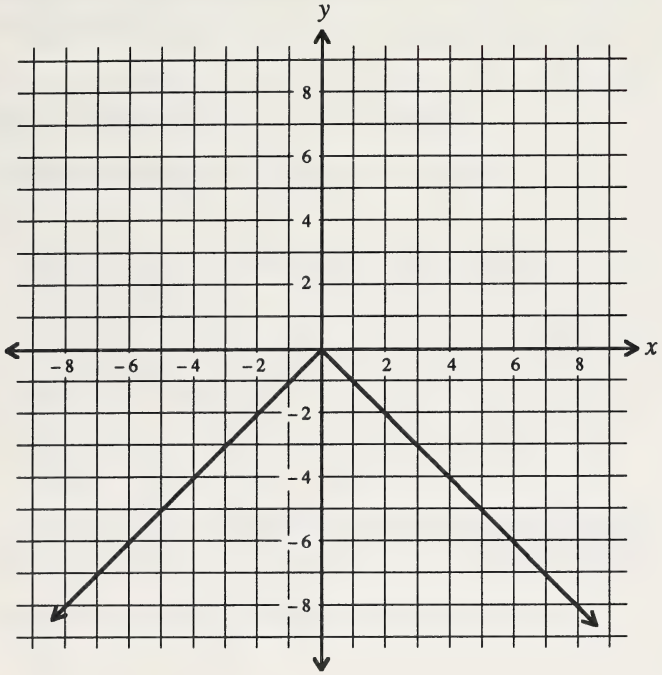
- $y=|2x|$
- $f(x)=5-|x|$
- $g(x)=|2x-4|$
- $y=|-5x|-4$

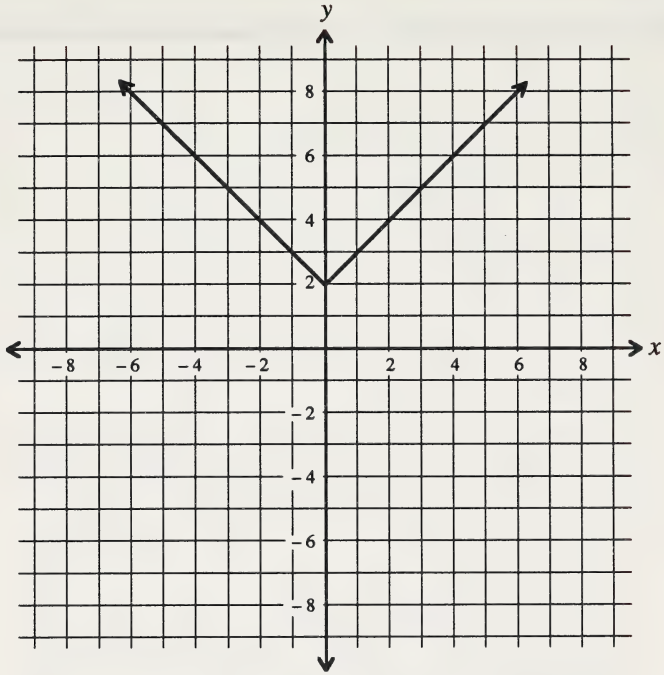
Graphs of absolute-value functions have a characteristic shape that can be viewed as two parts; one of the parts is a mirror image of the other. The mirror would be a vertical line at the point where the parts meet.

The expression $|-5|$ means the absolute value of -5 or the magnitude of -5.

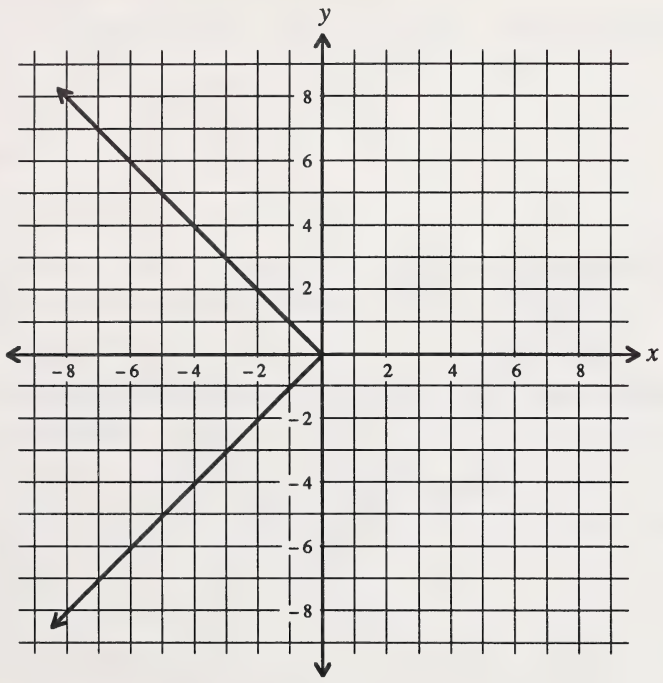
$$|x|=\begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

The following are examples of the graphs of absolute-value functions.

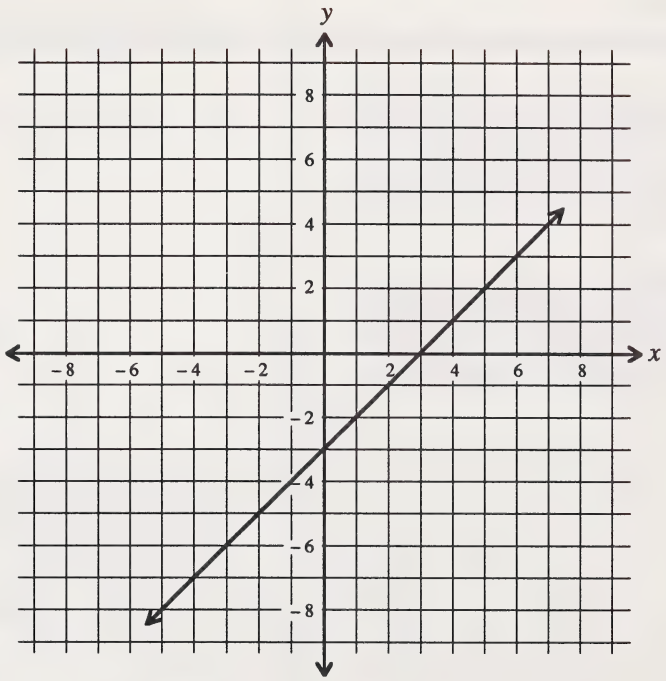




The following are examples of graphs which are not absolute-value functions.



This is not a function.



This is a linear function.

Square-Root Functions

Square-root functions have characteristic equations of the form $y=\sqrt{x}$, **where** $x \geq 0$.

The value of x must be greater than or equal to zero because you cannot take the square root of a negative number.

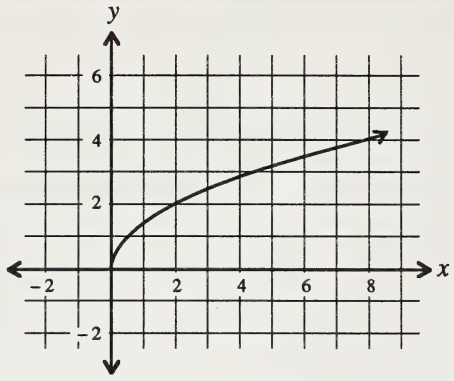
The following are examples of square-root functions.

- $y=\sqrt{2x}$, $x \geq 0$
- $f(x)=\sqrt{6x}-2$, $x \geq 0$
- $g(x)=4\sqrt{x-7}$, $x \geq 7$
- $y=\sqrt{8-x}$, $x \leq 8$

The graphs of square-root functions have a characteristic shape. The graph begins at some fixed point and continues in one direction.

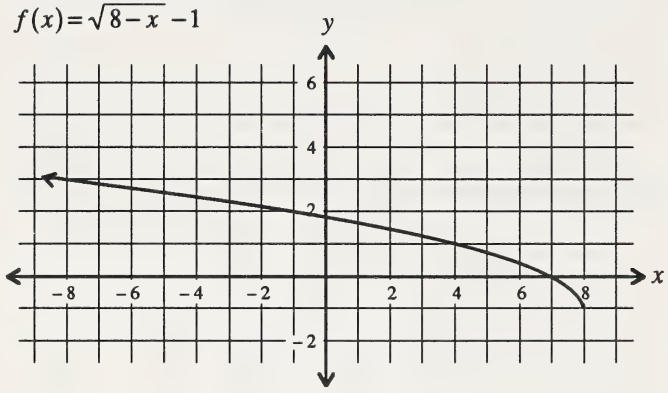
The following are examples of graphs of square-root functions.

- $y=\sqrt{2x}$



The domain of this function is $x \geq 0$, $x \in R$.
The range of this function is $y \geq 0$, $y \in R$.

- $f(x)=\sqrt{8-x}-1$



The domain of this function is $x \leq 8$, $x \in R$.
The range of this function is $y \geq -1$, $y \in R$.

Exponential Functions

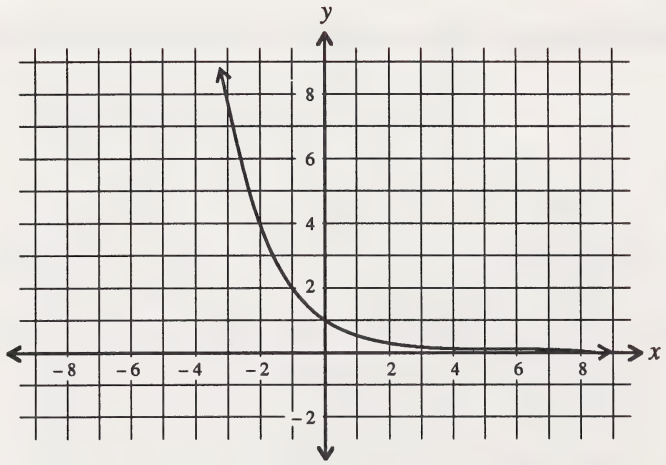
Exponential functions have the characteristic equation $y = n^x$ where $n \geq 0,\ n \in R$.

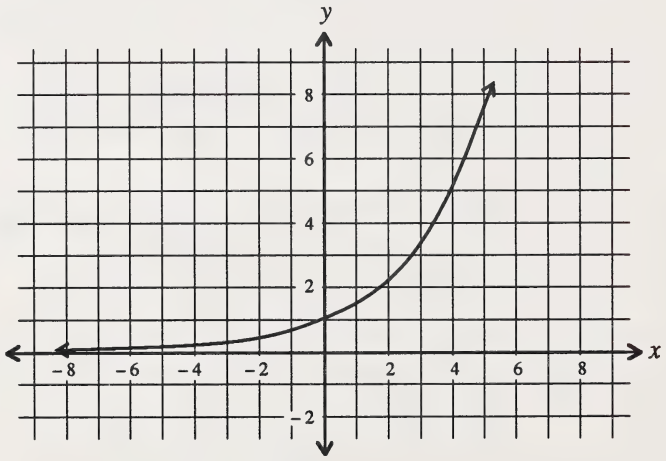
The following are examples of exponential functions.

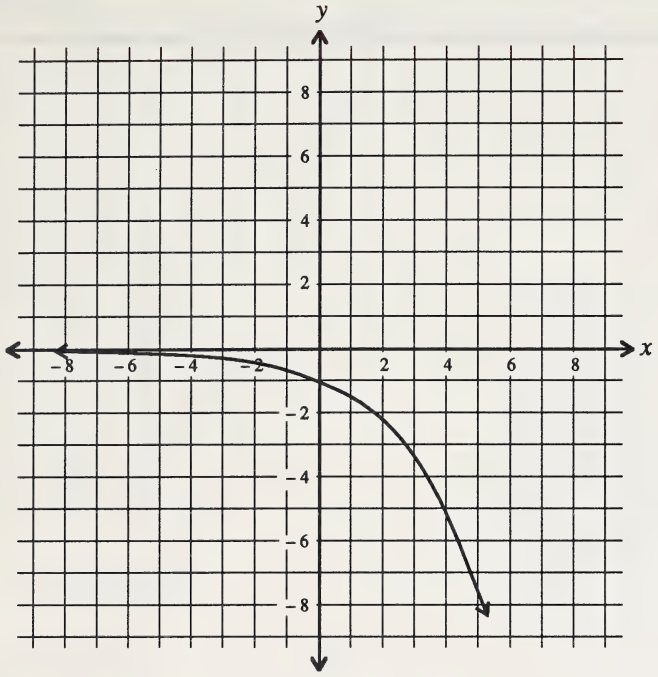
- $y = 3^x$
- $y = 2^x - 3$
- $f(x) = 2^{x+2}$
- $f(x) = \left(\frac{1}{2}\right)^{x+1}$
- $g(x) = -2^x$
- $g(x) = -3^{x-1} + 5$

The graphs of exponential functions have a characteristic shape. The graph increases in one direction and decreases in the other direction.

The following are examples of graphs of exponential functions.






y
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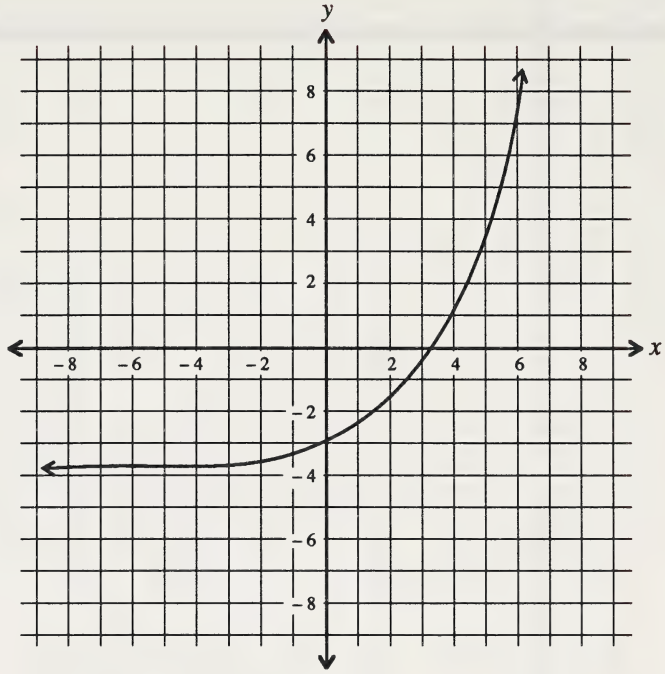
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Do all of questions 1 and 2 and do any two of questions 3 to 5.

1. From the following list, choose the polynomial function that matches with each type of function. Then make a table of values for the function using values of x from – 7 to 6 and draw the graph of each function.

$f(x)=2^{x+1}$ $\qquad$ $y=|x+3|-6$

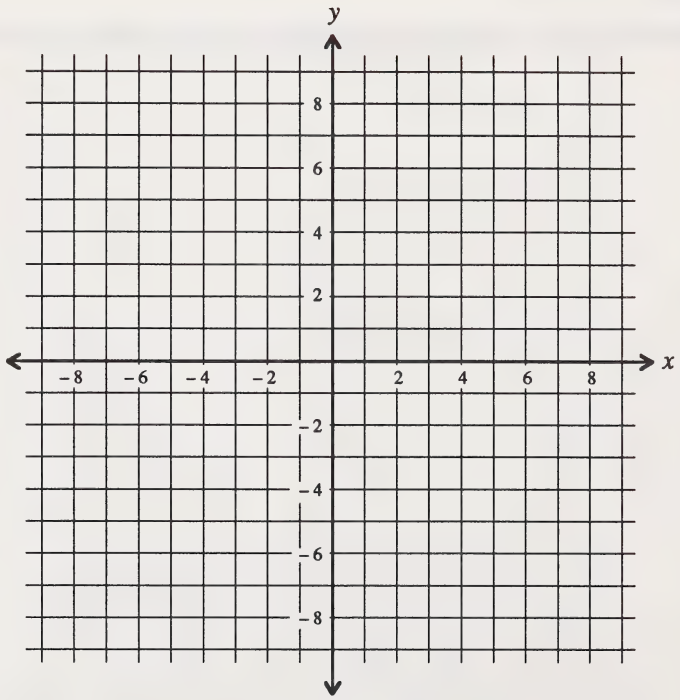
$g(x)=-2x+5$ $\qquad$ $f(x)=-\frac{1}{2}x^3+3$

$y=\frac{1}{2}x^2-3$ $\qquad$ $Q(x)=2\sqrt{3-x}$

$P(x)=\frac{1}{x}$ $\qquad$ $g(x)=x^2-x+3$

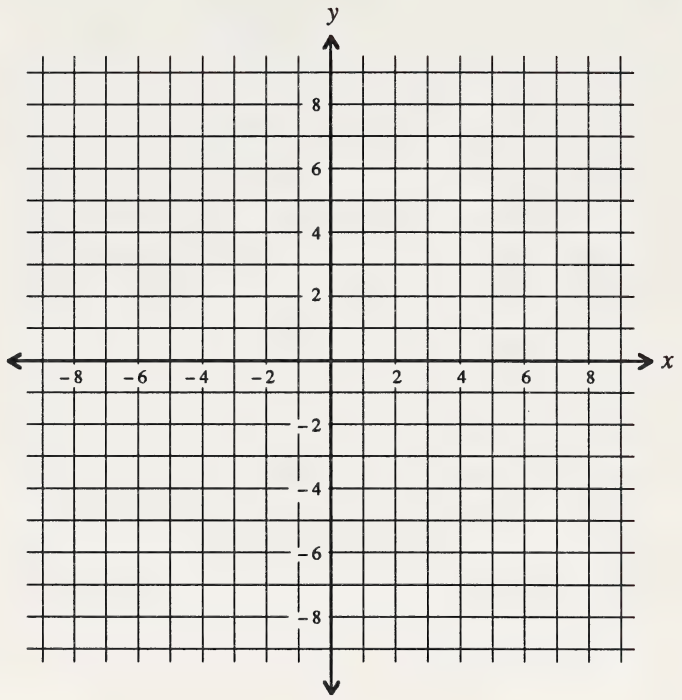
a. linear function _____

x	y



b. square-root function _____

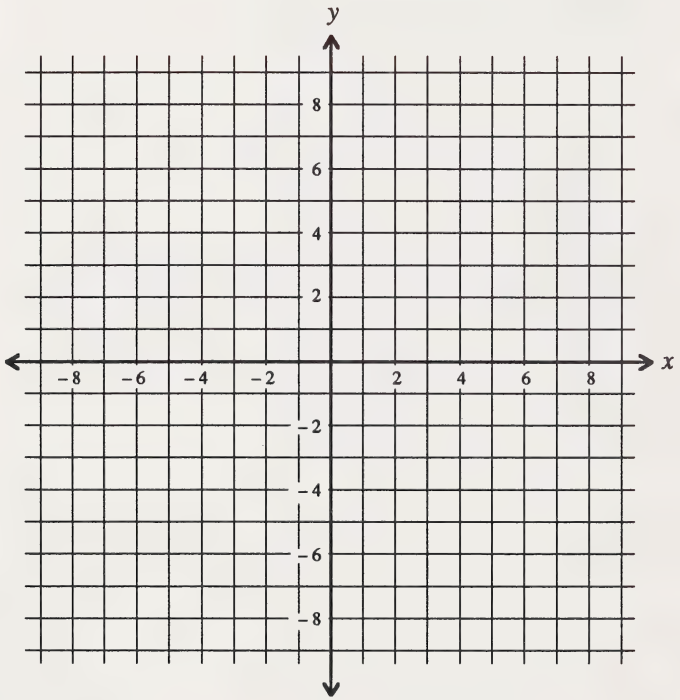
x	y



c. reciprocal function _____

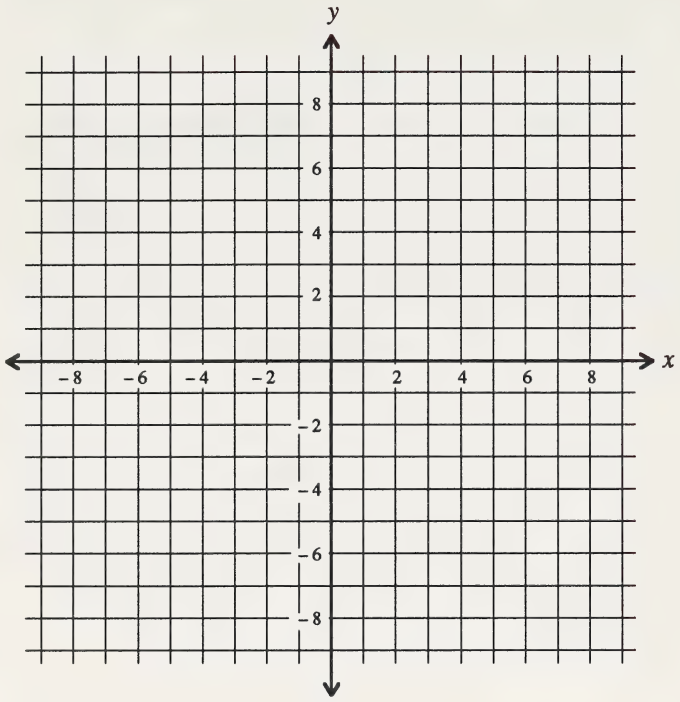
x	y
-6	
-5	
-4	
-3	
-2	
-1	
$-\frac{1}{2}$	
$-\frac{1}{3}$	
$-\frac{1}{4}$	

x	y
6	
5	
4	
3	
2	
1	
$\frac{1}{2}$	
$\frac{1}{3}$	
$\frac{1}{4}$	



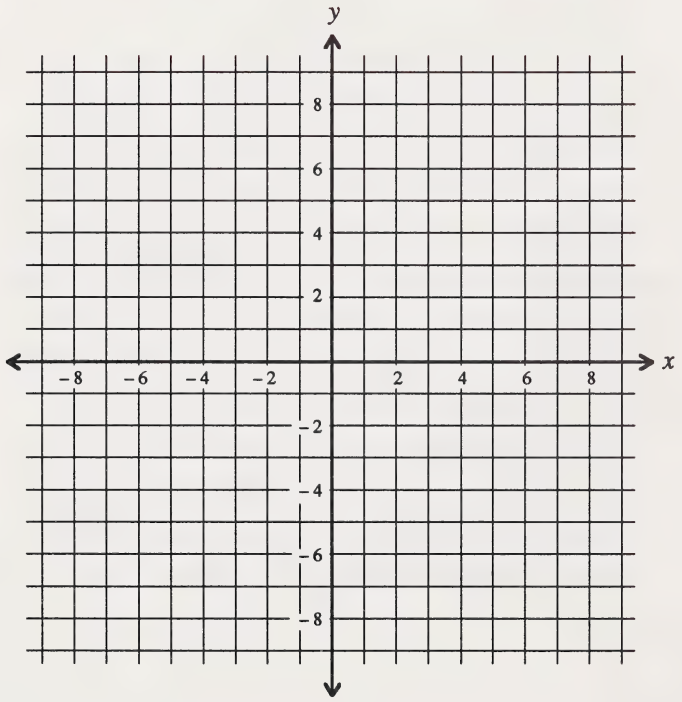
d. exponential function _____

x	y



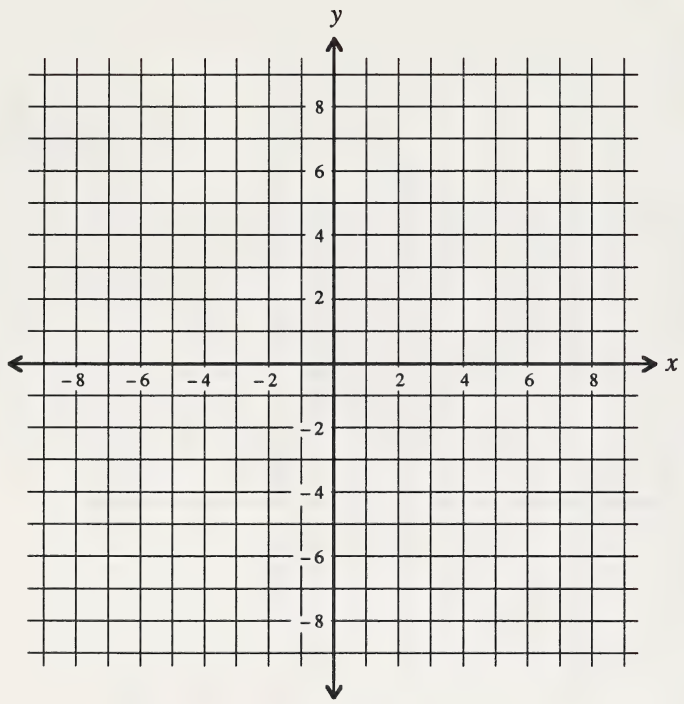
e. cubic function _____

x	y



f. absolute-value function _____

x	y



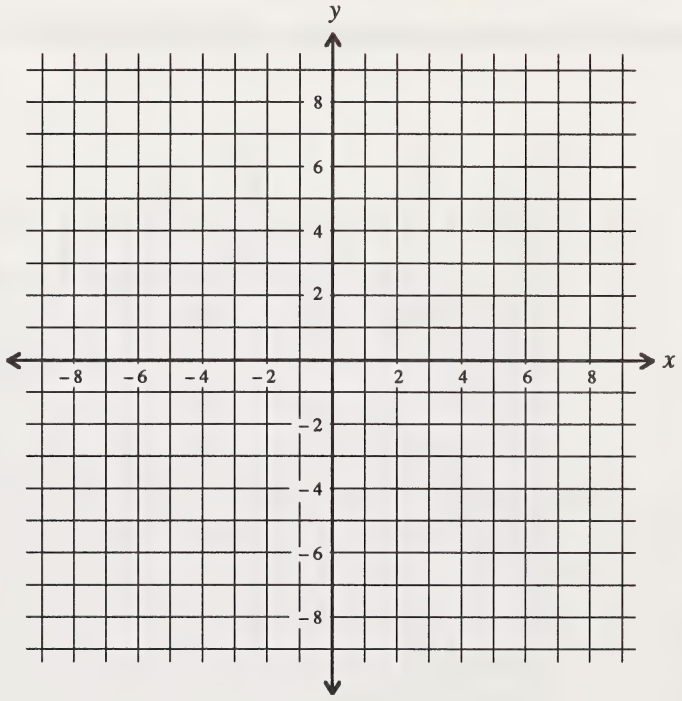
g. quadratic function

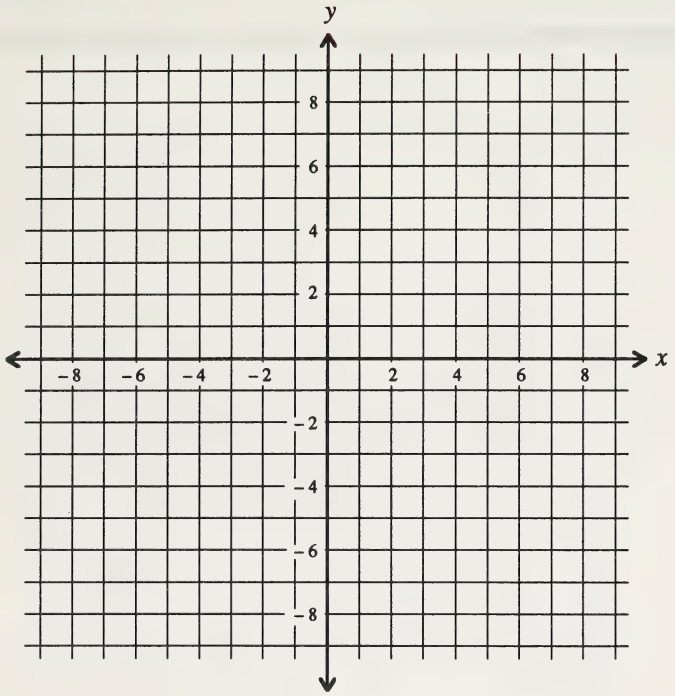
i. _____

x	y

ii. _____

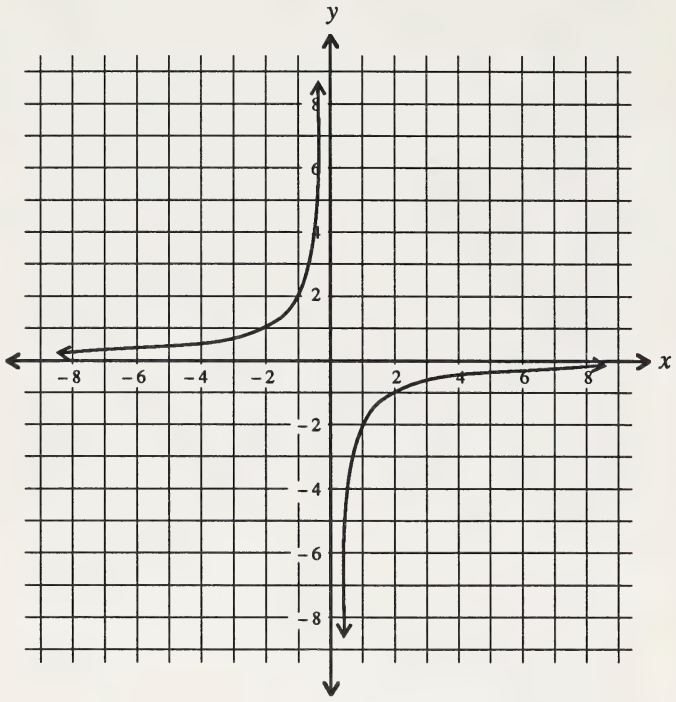
x	y





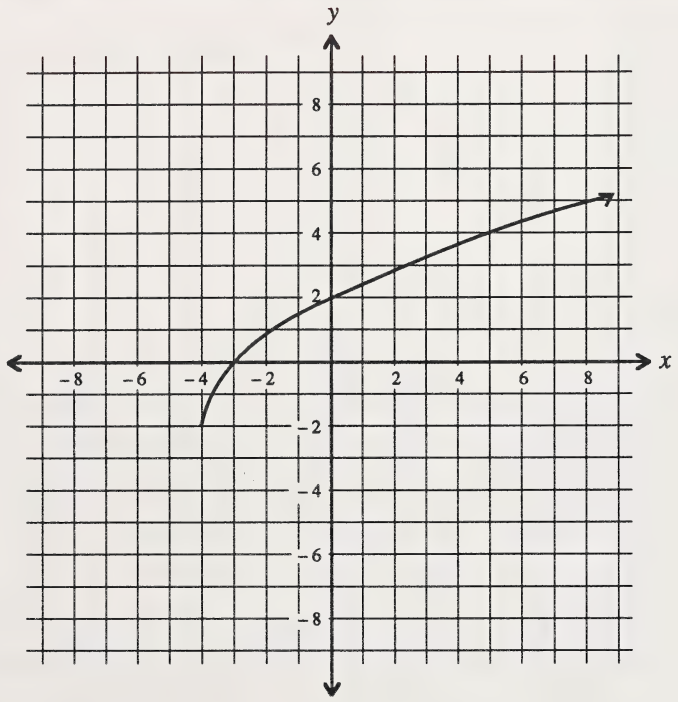
2. In the blank following each graph state the type of function (linear, quadratic, cubic, reciprocal, absolute-value, square-root, or exponential). If it is not a function, state the answer as **not a function**.

a.

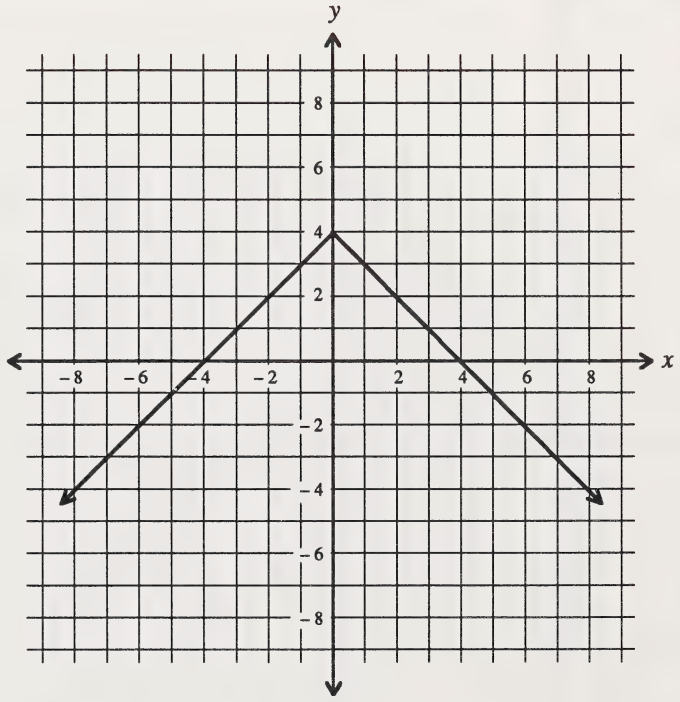


_____

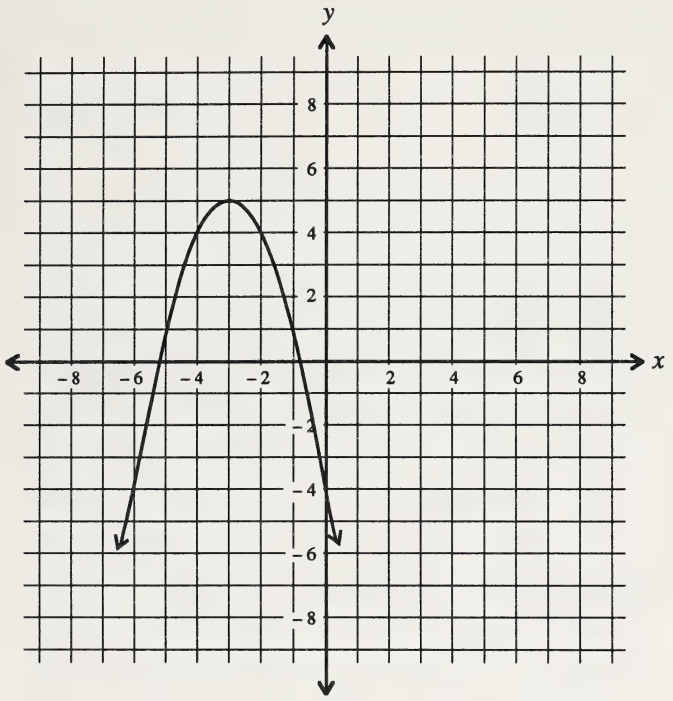
b.



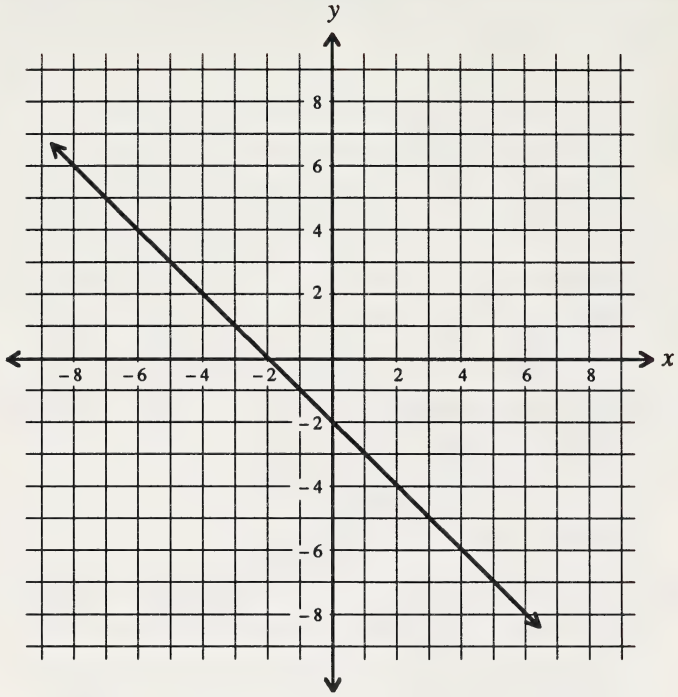
c.



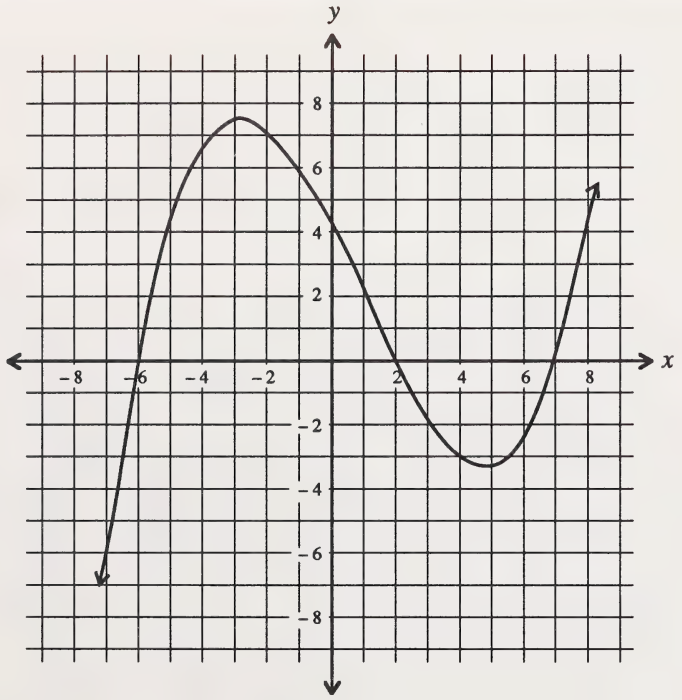
d.



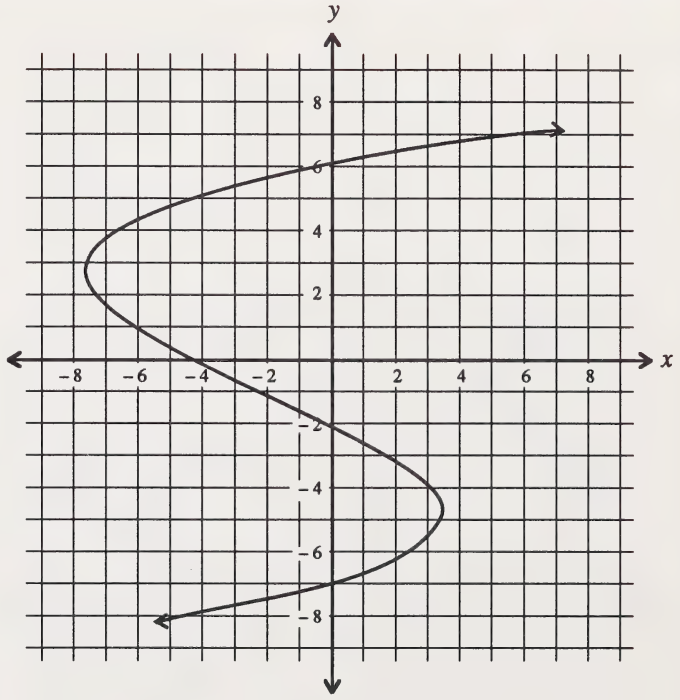
e.



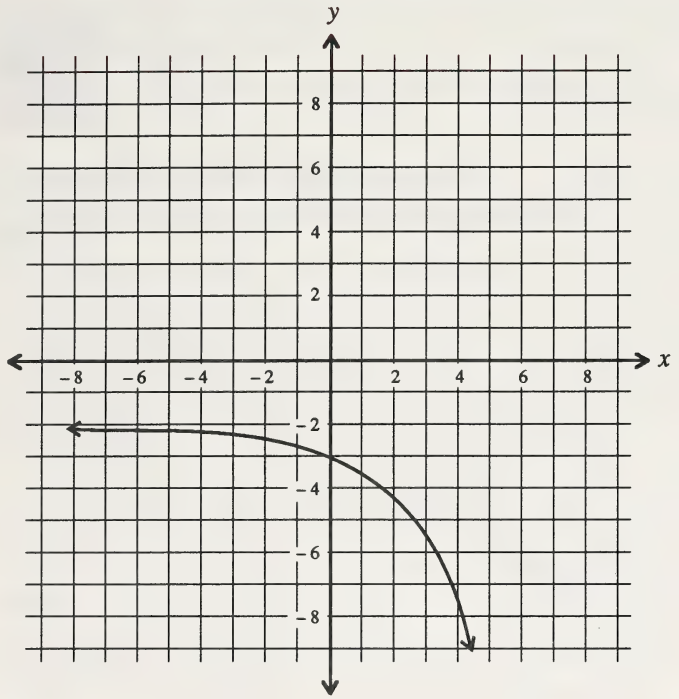
f.



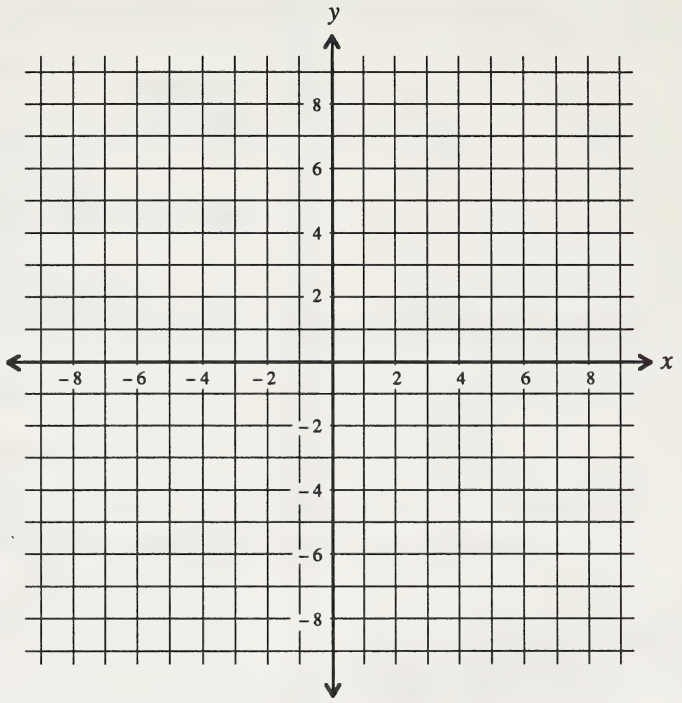
g.

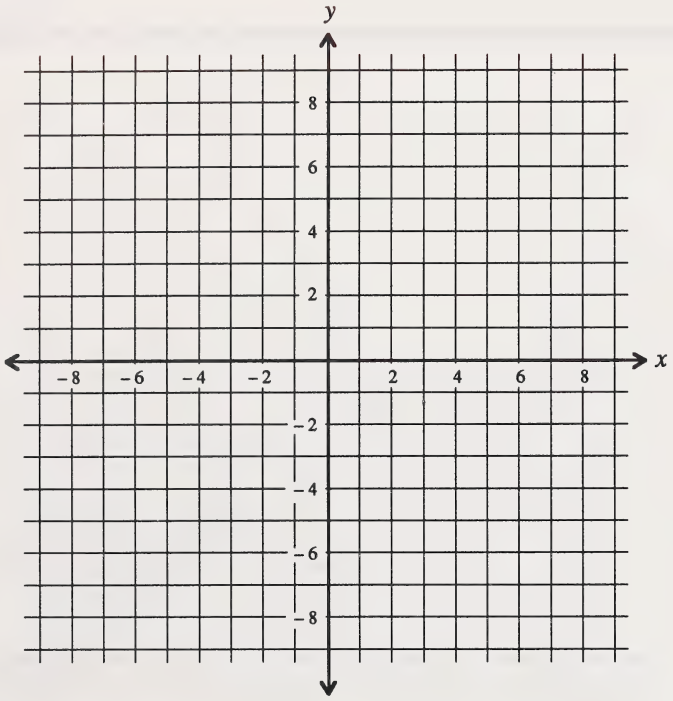


h.



3. Sketch the characteristic graphs of any quadratic functions. (Hint: There are two possibilities.)





4. Give the general equation for a reciprocal function.

5. Explain in words the difference between the graphs of an exponential function and a square-root function.

For solutions to **Activity 3**, turn to **Appendix A, Topic 2.**

Activity 4

Solve problems involving functions algebraically and by drawing, interpolating, extrapolating, interpreting, and analyzing graphs of functions.

In this section you will combine the knowledge of the previous sections and solve problems using a variety of methods.

Example 10

A steel worker earns $12.50 per hour. Write an equation in functional notation relating the earnings per week to the number of hours worked. Determine the earnings if the steel worker worked 39 h last week.

Solution:

Let $S(h)$ equal the wages per week. Let h equal the number of hours worked.

$$S(h) = (\$12.50)(h)$$

$$S(39) = (\$12.50)(39)$$
$$= \$487.50$$

The steel worker's earnings for the week were $487.50.

Example 11

The circumference of a circle varies with the radius. The relationship is as follows:

$$C = 2\pi r$$

- Write this relationship in functional notation.

Solution:

$f(r) = 2\pi r$

There are many other possible ways to write this function. Some are as follows:

$C(r) = 2\pi r$

$g(x) = 2\pi x$

- Calculate the circumference when the radius is 10 cm.

Solution:

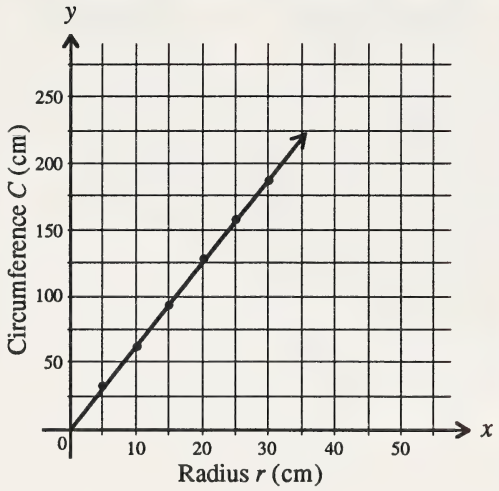
$$\begin{aligned} f(10\text{ cm}) &= 2\pi(10\text{ cm}) \\ &= 2(\pi)(10\text{ cm}) \\ &\doteq 62.8\text{ cm} \end{aligned}$$

- Draw the graph of this function for values of r to 20 cm.

Solution:

Make a table of values.

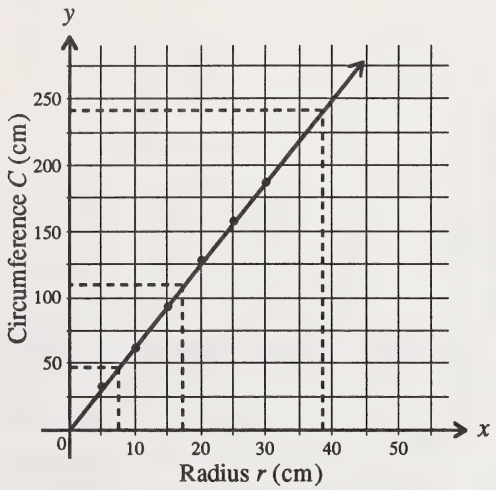
r (cm)	C (cm)
5	31.4
10	62.8
15	94.2
20	125.7
25	157.1
30	188.5



$\pi \doteq 3.141\,592\,654$

- From the graph determine C when r is 7.5 cm and 17.5 cm. Extrapolate the graph to find C when $r = 38$ cm.

Solution:



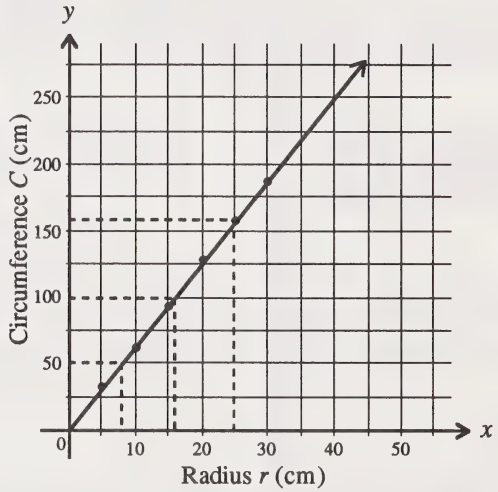
When $r = 7.5$ cm, $C \doteq 47$ cm.

When $r = 17.5$ cm, $C \doteq 110$ cm.

When $r = 38$ cm, $C \doteq 239$ cm.

- From the graph determine r when C is 50 cm, 100 cm, and 160 cm.

Solution:



When $C = 50$ cm, $r \doteq 8$ cm.

When $C = 100$ cm, $r \doteq 17$ cm.

When $C = 160$ cm, $r \doteq 25$ cm.

- What type of function does this graph represent?

Solution:

This graph represents a linear function.

- From the graph state the domain and the range.

  Solution:

  The domain is $r \geq 0,\ r \in R$. (You cannot have negative radii.)
  The range is $C \geq 0,\ C \in R$.

Example 12

The kinetic energy (measured in J) of an object is equal to one-half the mass of the object (in kg) times the velocity of the object (in m/s) squared.

- Write an equation for this function.

  Solution:

  $$KE = \frac{1}{2}mv^2$$

- Calculate the kinetic energy of an object for which mass is 500 kg and velocity is 20 m/s.

  Solution:

  $$KE = \frac{1}{2}(500\text{ kg})(20\text{ m/s})^2$$
  $$= \frac{1}{2}(500\text{ kg})(400\text{ m}^2/\text{s}^2)$$
  $$= 100\,000\text{ J}$$

- Draw a graph of this function when the mass is 500 kg.

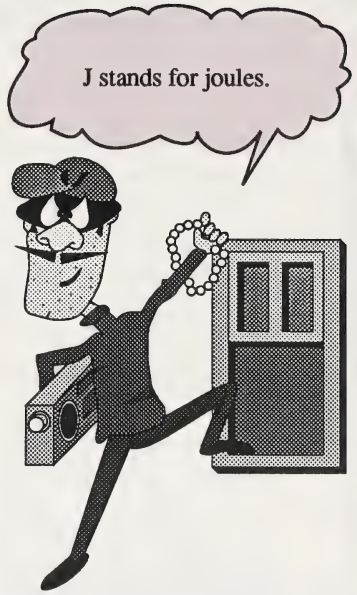
  Solution:

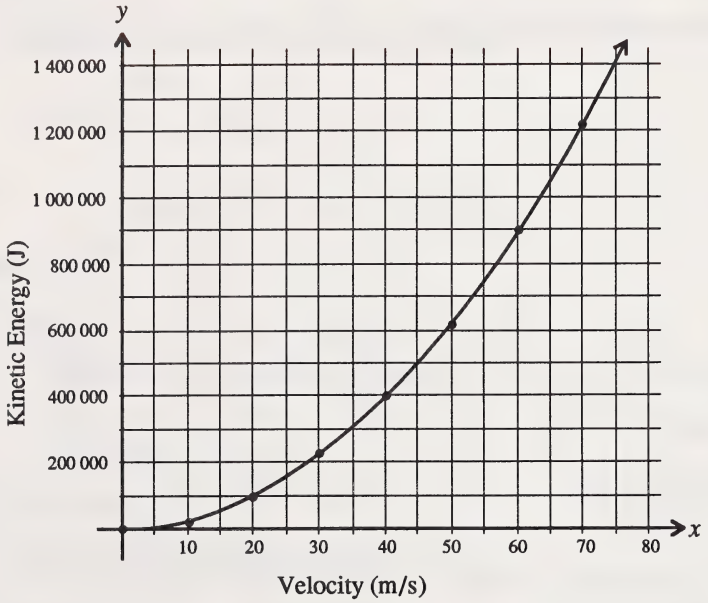
  Make a table of values.

Velocity	*KE*
10 m/s	25 000 J
20 m/s	100 000 J
30 m/s	225 000 J
40 m/s	400 000 J
50 m/s	625 000 J

Following is a sample calculation for $v = 30\text{ m/s}$.

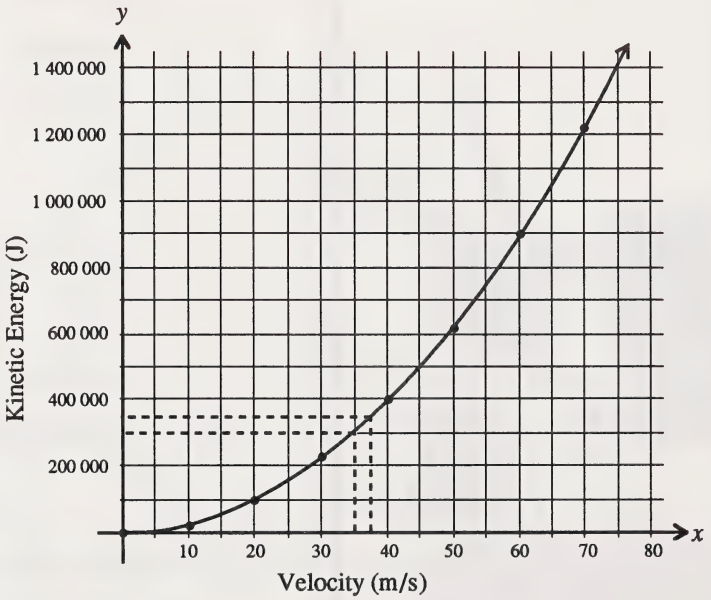
$$KE = \frac{1}{2}(500\text{ kg})(30\text{ m/s})^2$$
$$= \frac{1}{2}(500\text{ kg})(900\text{ m}^2/\text{s}^2)$$
$$= 225\,000\text{ J}$$





- From the graph determine the following:
  - KE when velocity is equal to 35 m/s
  - velocity when KE is 350 000 J

Solution:



From the graph $KE \doteq 300\,000$ J when $v = 35$ m / s.

From the graph $v \doteq 37.5$ m / s when $KE = 350\,000$ J.

- What type of function is this?

Solution:

This is a quadratic function.

- State the domain and the range.

  Solution:

  The domain is $v \geq 0$ m/s, $v \in R$.
  The range is $KE \geq 0$ J, $KE \in R$.

The examples demonstrate the basic steps required to solve problems involving functions. The exercise that follows contains many questions involving a variety of functions.

Do any three of the following six questions. Graph paper is provided in **Appendix B.**

1. A force acting on an object produces an acceleration. The relationship between the force, the acceleration of the object, and the mass of the object is given by $a = \frac{F}{m}$.

   F = force (newtons)

   m = mass (kg)

   a = acceleration $\left(\text{m/s}^2\right)$

   a. Draw the graph of acceleration versus force. Place force on the horizontal axis for a fixed mass of 10 kg.

   b. Interpolate from the graph the acceleration when $F = 650$ N.

   c. Determine the force when the acceleration is 100 m/s^2.

2. The area of a triangle is equal to one-half the base times the height.

   a. Write an expression for this function.

   b. Evaluate the expression for a triangle where the base equals 15 cm and the height equals 23 cm.

   c. Draw the graph of the function for triangles with a constant base of 15 cm.

   d. From the graph, what is the area of a triangle with a base of 15 cm and a height of 43 cm?

   e. What type of function does this graph represent?

3. The power in an electric circuit is given by $P = \frac{V^2}{R}$, where P = power (watts), V = voltage (volts), and R = resistance (ohms).

   a. What type of function is this?

   b. Draw the graph of this function for a fixed resistance of 10 Ω.

   c. Interpolate the power when $V = 70$ V.

Ω is the symbol for ohms.

4. The intensity of light at a point varies inversely as the square of the distance from the source. The equation relating intensity and distance is as follows:

$I = \frac{k}{d^2}$, where I = light intensity at the point in lumens,
d = distance in metres, and
k = a constant

(For the purpose of this question let $k = \frac{1000 \text{ lm}}{\text{m}^2}$, so that the units of I are lumens.)

a. Evaluate the function for a distance of 15 m.

b. Draw the graph of the function. (Use values of d from 1 m to 15 m.)

c. From the graph determine the intensity at a distance of 2.50 m from the source.

d. From the graph determine the distance from the source at which the intensity is 500 lumens (1m).

5. If a ball is thrown upwards with a velocity of 25 m/s, the distance h above the ground is given by $h = -5t^2 + 25t$, where t is time in seconds and h is height in metres.

a. Draw the graph of this function.

b. According to the graph, what is the maximum height of the ball?

c. From the graph determine the total time the ball is in the air.

d. From the graph determine the two times when the height of the ball is 25 m.

6. A local fund-raiser has $9000 in prizes guaranteed. Only 5000 tickets are printed and they sell for $5.00 each.

a. Draw the graph of money raised versus number of tickets sold.

b. According to the graph, how many tickets must be sold to raise
$5000 after the prizes are paid?

c. What is the most money that the fund-raiser can raise after the prizes have been given out?

d. How many tickets must be sold to break even?

e. What is the domain and the range?

For solutions to **Activity 4**, turn to **Appendix A, Topic 2.**

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

If you had some difficulties with this unit and have access to a videocassette recorder (VCR), you may wish to view the video titled **Introduction to Functions**. This video is program 2 in the *Math Sense*[1] series and will reinforce some of the difficult concepts in the study of functions.

All functions are relations.

When is a relation not a function? How can you tell? There are two ways to tell if a relation is not a function.

- Are there two points with the same x-coordinate?
- Does a vertical line pass through more than one point on the graph?

If the answer to either of the questions is yes, the relation is not a function.

[1] *Math Sense* is a title of ACCESS Network.

Functional notation is used to simplify and make algebraic statements more meaningful.

The expression $f(x)$ is read ***f* of *x*** and is used in the following way.

Instead of writing $y=2x^2+3x+2$, you write $f(x)=2x^2+3x+2$. This notation allows for evaluation of the function at any value of x. For example,

$$f(x)=2x^2+3x+2$$
$$f(-2)=2(-2)^2+3(-2)+2$$
$$f(-2)=8-6+2$$
$$f(-2)=4$$

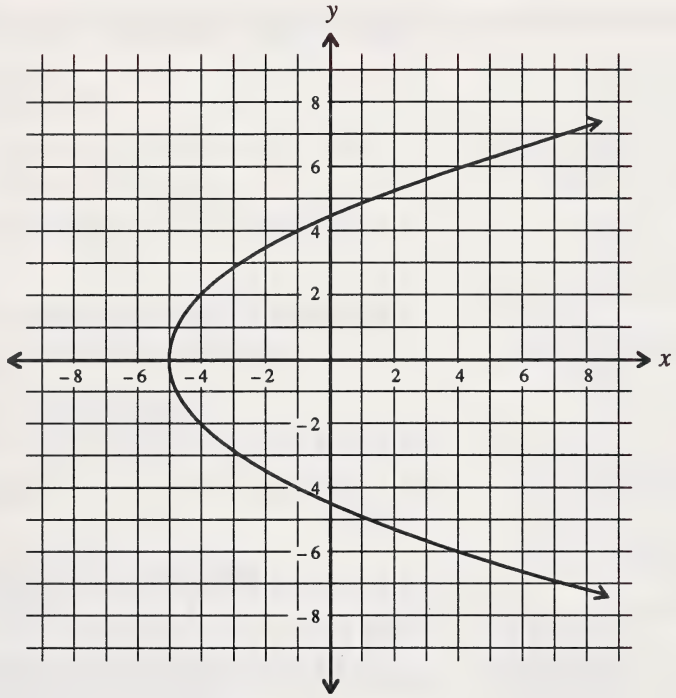
Try the following questions.

1. Determine whether each relation is a function.

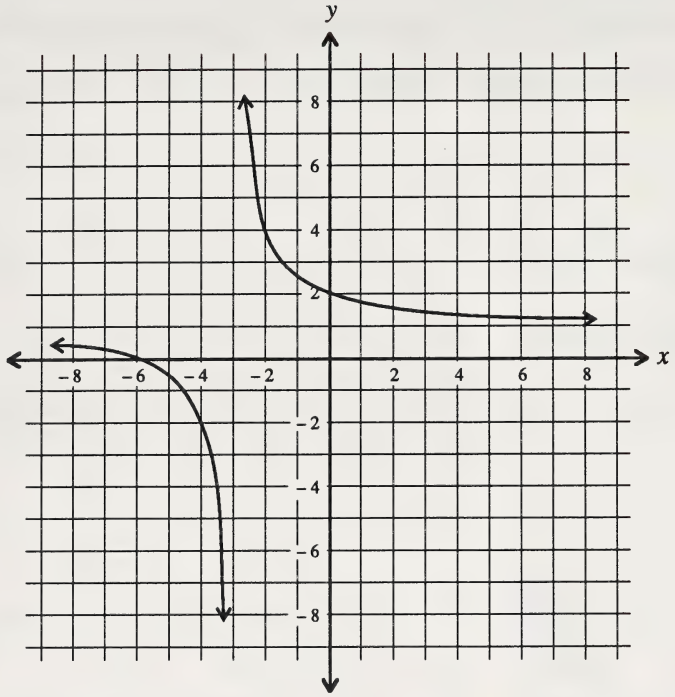
   a. $y=3-x$

   b. $\{(0,-1),\ (2,5),\ (2,-5),\ (3,2)\}$

c.



d.



2. Evaluate the following.

a. $f(-3)$ when $f(x)=x^2-20$

b. $g(5)$ when $g(x)=x^3-x^2+3x$

c. $P(0)$ when $P(x)=3x^4-5x^3+2$

For solutions to **Extra Help**, turn to **Appendix A, Topic 2.**

Extensions

Composition of Functions

Functions can be defined in terms of other functions. This idea is shown in the following example.

Let $f(x)=2x^2-3x$ and $g(x)=x+2$.

Then, $f(g(x))=2(x+2)^2-3(x+2)$ (Each x in $2x^2-3x$ is replaced by $x+2$.)

$=2(x^2+4x+4)-3(x+2)$

$=2x^2+8x+8-3x-6$

$=2x^2+5x+2$

What is $g(f(x))$?

$g(f(x))=(2x^2-3x)+2$ (Replace the x in $x+2$ with $2x^2-3x$.)

$=2x^2-3x+2$

The notation used is $(f\circ g)(x)=f(g(x))$ and it is read as **the composition of f and g**. Note that in general $f(g(x))\neq g(f(x))$.

The functional mapping of g is performed first, and then the mapping of f is performed.

Try the following questions.

For each pair of functions, find $f(g(x))$ and $g(f(x))$.

1. $f(x)=x^2-7x,\ g(x)=5x+1$

2. $f(x)=3x,\ g(x)=x^2-1$

3. $f(x)=\frac{1}{2x-1},\ g(x)=5x+2$

For solutions to **Extensions**, turn to **Appendix A, Topic 2.**

Topic 3 Transformations of Functions

Introduction

A large tree is cut down and sawed into lumber. The lumber is used to build a house and a desk. The transformation of the tree into the house and the desk is dramatic and it is often difficult to see the tree once it is changed into the finished products.

A small tree grows into a large tree. It is relatively easy to see the small tree in the large tree since only the dimensions have changed.

You will study transformations of functions in ways that allow you to see the original in the transformation.

If you have access to a videocassette recorder (VCR), you may wish to view the programs titled **Vertical Translations of Functions** and **Horizontal Translations of Functions.** These two videos are programs 3 and 4 in the *Math Sense*[1] series and will give you a good overview of vertical and horizontal translation of functions.

[1] *Math Sense* is a title of ACCESS Network.

What Lies Ahead

Throughout the topic you will learn to

1. investigate and describe the transformation effects of the parameters a and b in $y = f(x-a)+b$ on the graph of $y = f(x)$
2. investigate and describe the transformation effects of the parameter c in $y = cf(x)$ on the graph of $y = f(x)$
3. describe and draw the graphs of $y = cf(x-a)+b$

Now that you know what to expect, turn the page to begin your study of transformations of functions.

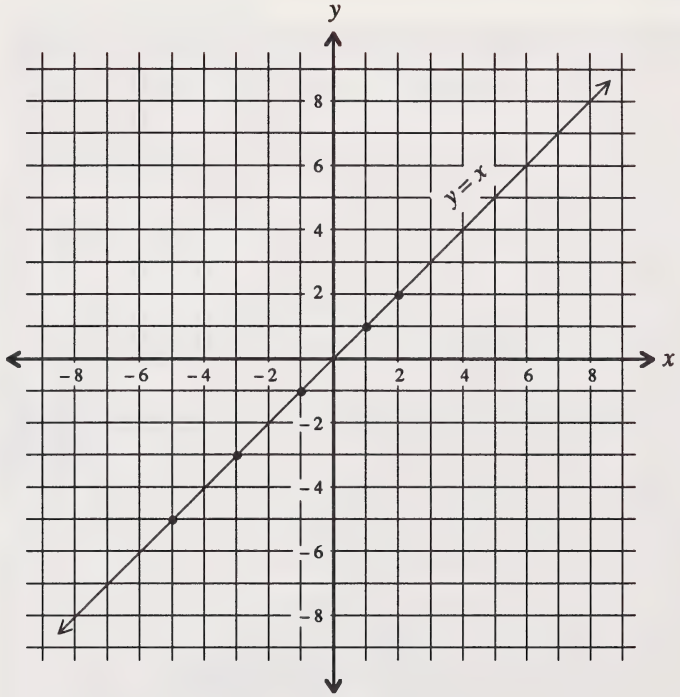
Exploring Topic 3

Activity 1

Investigate and describe the transformation effects of the parameters a and b in $y = f(x-a)+b$ on the graph of $y = f(x)$.

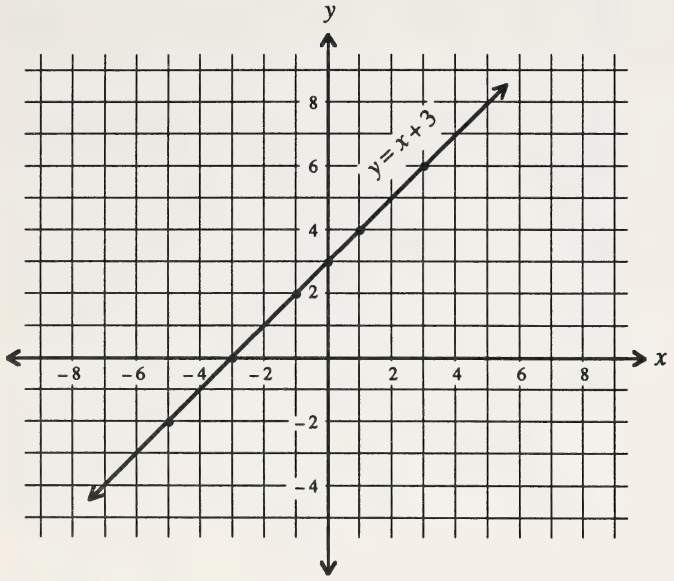
The notation in the previous statement will be developed and explained as this topic unfolds. The reason for using this notation will become clearer as you work through this section. You will begin with the transformation of $y = f(x)$ to $y = f(x)+b$. Examine the graph of $y = x$. A table of values is made and the graph of this function is plotted.

x	y
−5	−5
−3	−3
−1	−1
0	0
1	1
2	2

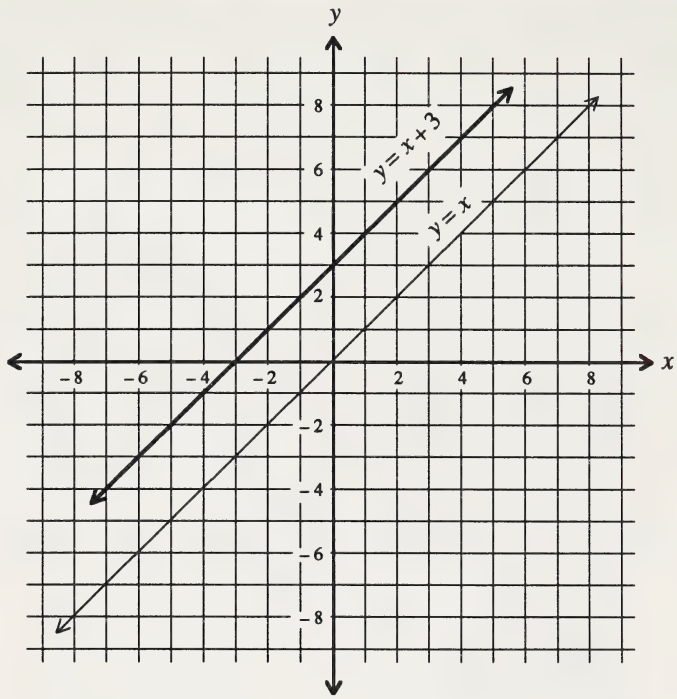


Make a small change to the function $y = x$ by adding 3 to get $y = x + 3$. A table of values is made and the graph is drawn.

x	y
-5	-2
-3	0
-1	2
0	3
1	4
3	6



Both graphs are plotted on the following grid.

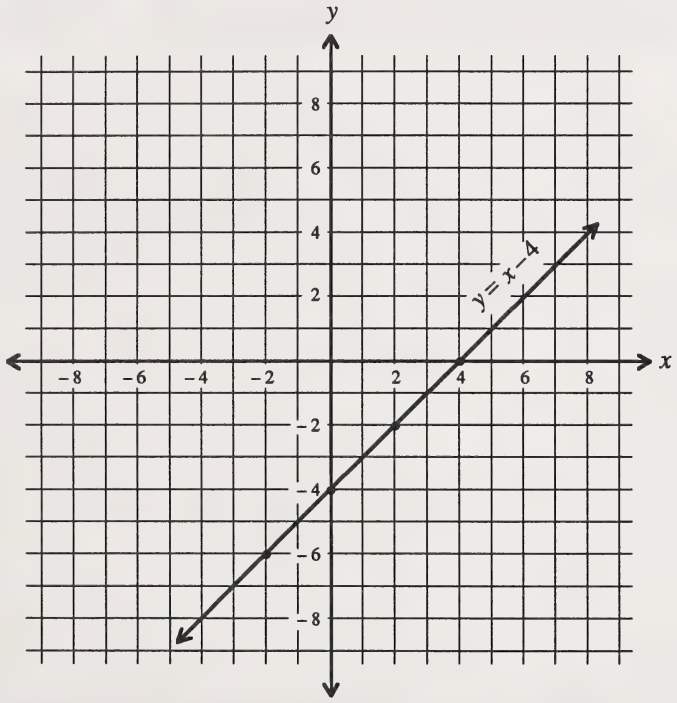


The effect of transforming $y = x$ to $y = x + 3$ is to move the graph of $y = x$ vertically upwards three units.

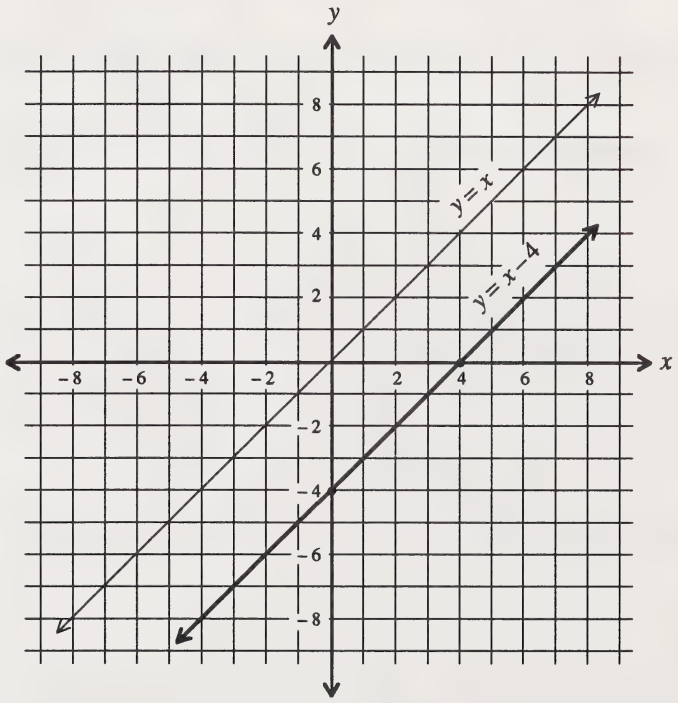
What effect will adding -4 to the function $y = x$ have on the graph?

Draw the graph of $y = x - 4$.

x	y
-2	-6
0	-4
2	-2
4	0
6	2
8	4

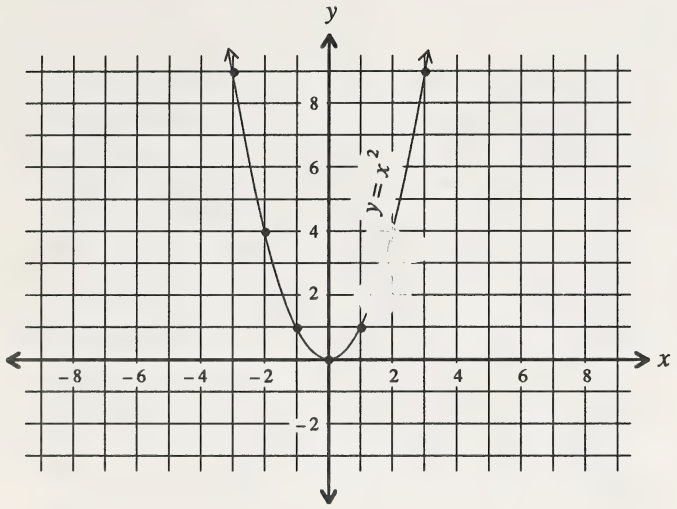


The effect of adding -4 is to move the graph vertically downwards four units. The graphs of $y = x$ and $y = x - 4$ are plotted as follows.



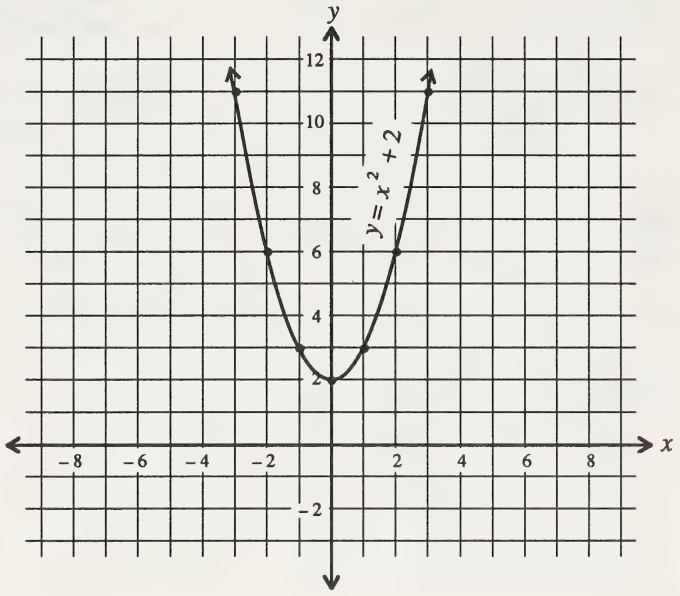
What are the effects of changing b in quadratic functions? Look at the graph of $y = x^2$.

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

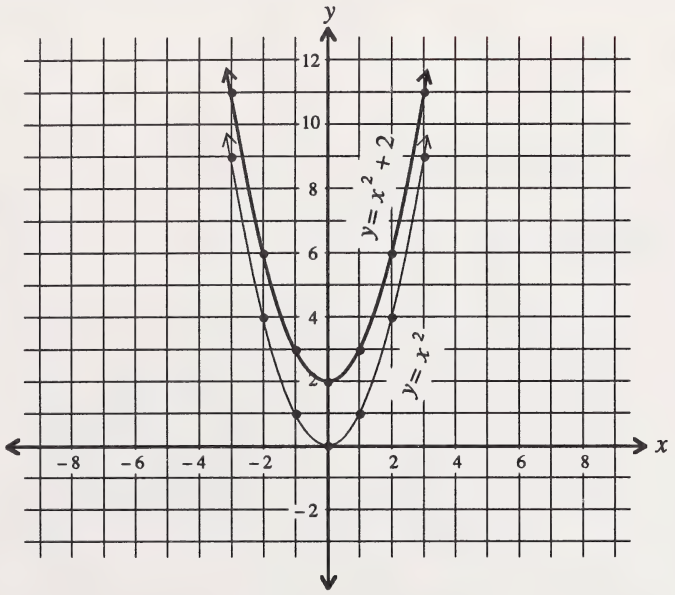


Now add 2 to the function to get $y = x^2 + 2$ and graph the function.

x	y
-3	11
-2	6
-1	3
0	2
1	3
2	6
3	11



Both graphs are plotted on the following grid.

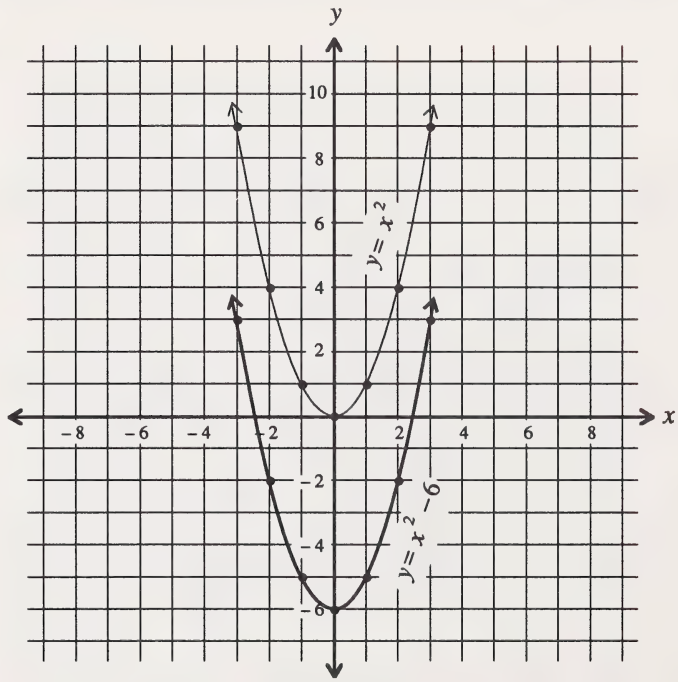


The transformation from $y = x^2$ to $y = x^2 + 2$ causes the graph to move upwards two units.

When $y = f(x)$ is transformed to $y = f(x) + b$, the graph of $y = f(x)$ moves vertically upwards b units if b is a positive number.

What do you predict will occur to the graph of $y = x^2$ if b is a negative number? If you said the graph will move vertically downwards b units, you are correct!

The graphs of $y = x^2$ and $y = x^2 - 6$ are as follows.



When $y = f(x)$ is transformed to $y = f(x) + b$, the graph of $y = f(x)$ moves downwards b units if b is a negative number.

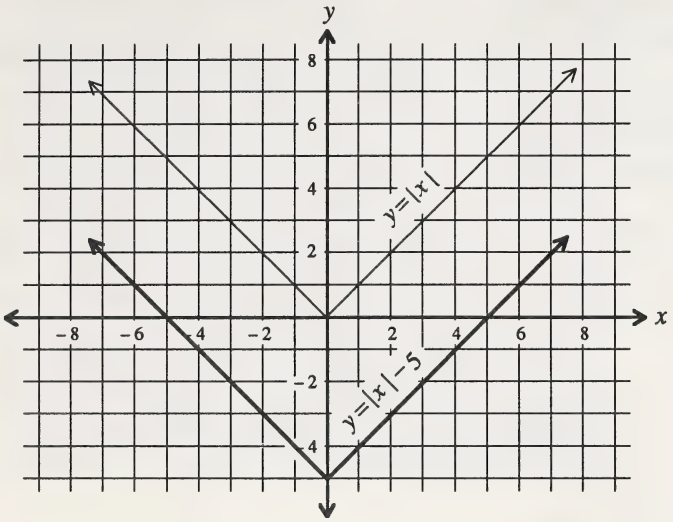
In general when $y = f(x)$ is transformed to $y = f(x) + b$, the graph moves upwards b units when b is a positive number and downwards b units when b is a negative number.

Example 1

- Without graphing, predict the effect of the transformation of $y = |x|$ to $y = |x| - 5$.

Solution:

According to the general statement, when $y = f(x)$ is transformed to $y = f(x) + b$, the graph will move downward b units when b is a negative number. Therefore, the graph of $y = |x| - 5$ will be the graph of $y = |x|$ moved down five units. Both graphs are shown.



- What do you predict the graph of $y = |x| + 7$ would look like?

Solution:

The graph of $y = |x| + 7$ will be the graph of $y = |x|$ shifted seven units upwards.

Examine the effects of the transformation of $y = f(x)$ to $y = f(x - a)$ by examining the function $y = x^2$ and its transformations.

Example 2

Draw the graphs of $y = x^2$, $y = (x-6)^2$, and $y = (x+6)^2$ on the same grid.

Solution:

Make a table of values for each function, then plot the functions.

The following are sample calculations:

$x = -10$,	$y = (-10)^2$	$y = (-10-6)^2$	$y = (-10+6)^2$
	$= 100$	$= (-16)^2$	$= (-4)^2$
		$= 256$	$= 16$

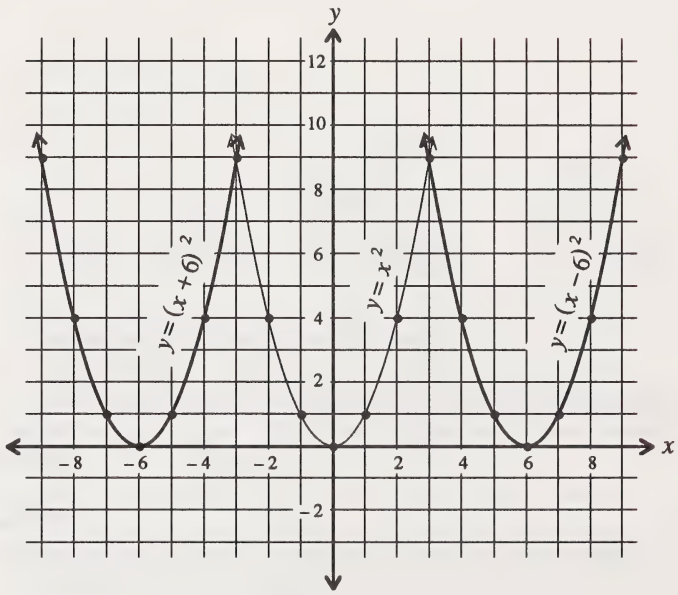
$x = 5$, $y = (5)^2$ $y = (5-6)^2$ $y = (5+6)^2$

$= 25$ $= (-1)^2$ $= (11)^2$

$= 1$ $= 121$

x	$y = x^2$	$y = (x-6)^2$	$y = (x+6)^2$
−10	100	256	16
−9	81	225	9
−8	64	196	4
−7	49	169	1
−6	36	144	0
−5	25	121	1
−4	16	100	4
−3	9	81	9
−2	4	64	16
−1	1	49	25
0	0	36	36
1	1	25	49
2	4	16	64
3	9	9	81
4	16	4	100
5	25	1	121
6	36	0	144
7	49	1	169
8	64	4	196
9	81	9	225
10	100	16	256

The graphs of the three functions are as follows:



The graph of $y = (x-6)^2$ is the graph of $y = x^2$ moved six units to the **right**.

The graph of $y = (x+6)^2$ is the graph of $y = x^2$ moved six units to the **left**.

Examine the previous statements and graphs to develop a standard by which to graph similar functions.

Hopefully you noticed that the direction the graph of $y = x^2$ moves is opposite to the sign of the number a in the transformation.

To correct this (moving in the wrong direction), the general statement of the transformation is changed to $y = f(x-a)$ rather than $y = f(x+a)$.

If you take the transformation as $y = f(x-a)$ and look at the graph, you will see something interesting. The graph moves in the correct direction with respect to the sign.

For example, $y = x^2$ is transformed to $y = (x-6)^2$. Here the value of a is $+6$ since $y = \big(x-(+6)\big)^2$ is equal to $y = (x-6)^2$. The graph moves to the right when a is positive.

This is also true when $y = x^2$ is transformed to $y = (x+6)^2$. Here the value of a is -6 since $y = \big(x-(-6)\big)^2$ is equal to $y = (x+6)^2$. The graph moves to the left when a is negative.

In general when $y = f(x)$ is transformed to $y = f(x-a)$, the graph moves to the right when a is positive and to the left when a is negative.

Example 3

Without graphing, predict the effect of the transformation of $y = |x|$ to $y = |x+7|$.

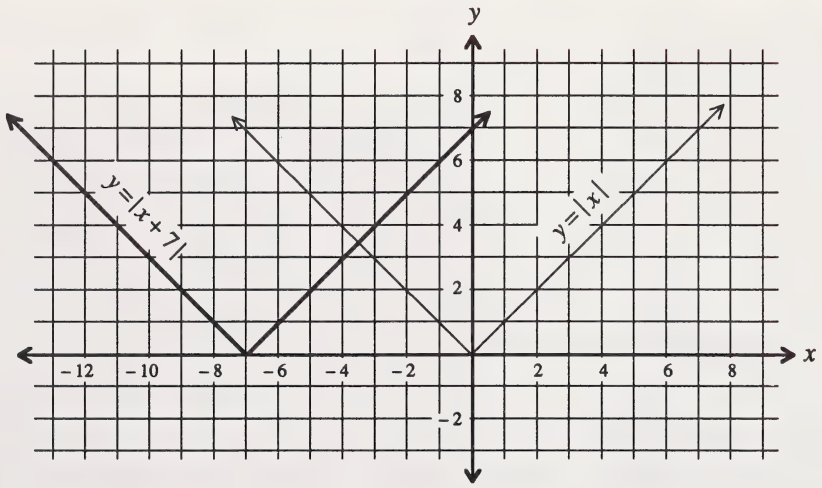
Solution:

According to the general statement that $y = f(x)$ is transformed to $y = f(x-a)$, the value of a is negative in the transformation in this question. Therefore, the graph will move to the left seven units. You can see this if you rewrite the equation.

The function $y = |x|$ is transformed to $y = |x-(-7)|$.

The general form for $y = (x-6)^2$ is $y = (x-a)^2$, where $a = 6$.

The graphs of the functions are as follows.



Example 4

Predict the effect of the transformation of $y = x^2$ to $y = (x-3)^2 - 5$.

Solution:

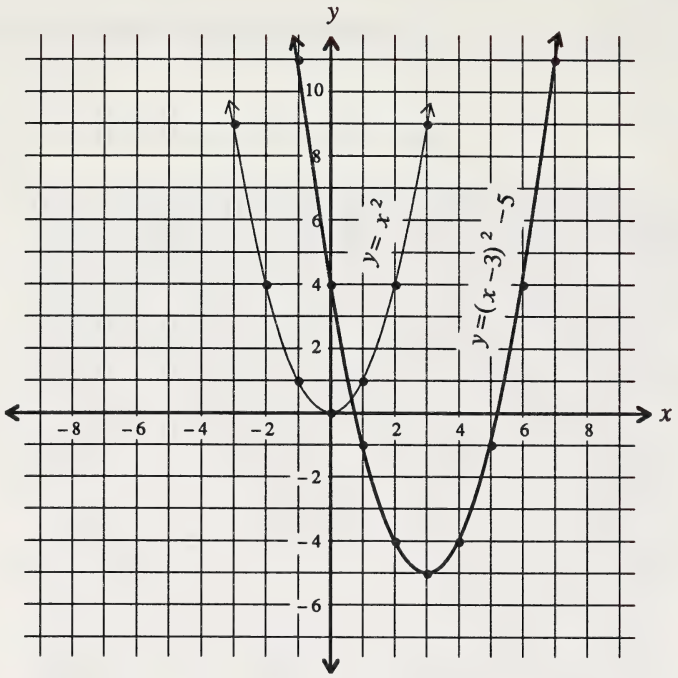
$b = -5$

$a = +3$

Therefore, the graph will move five units vertically down and three units to the right.

Check: The following are the graphs of $y = x^2$ and $y = (x-3)^2 - 5$.

x	$y = x^2$	$y = (x-3)^2 - 5$
-4	16	44
-3	9	31
-2	4	20
-1	1	11
0	0	4
1	1	-1
2	4	-4
3	9	-5
4	16	-4
5	25	-1
6	36	4
7	49	11

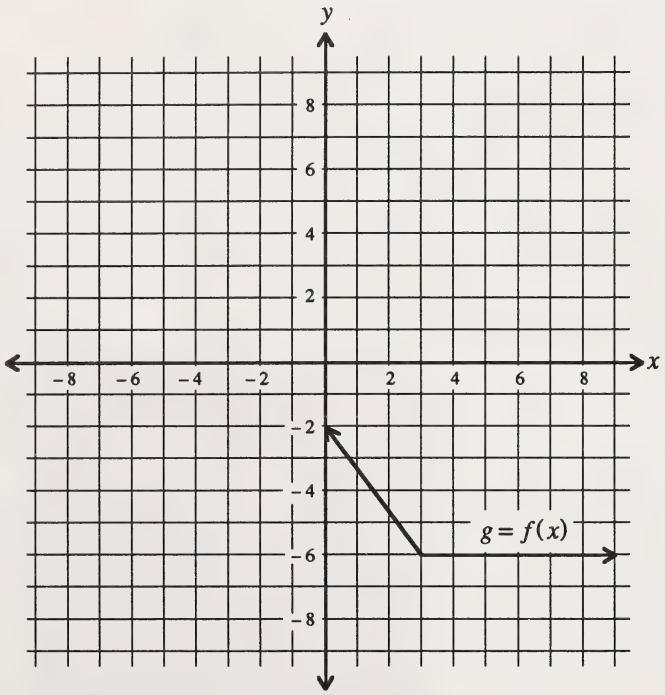


You can put both generalizations of transformations together into one statement.

In general when $y = f(x)$ is transformed to $y = f(x-a)+b$, the graph will move b units upwards when b is positive, b units downwards when b is negative, a units to the right when a is positive, and a units to the left when a is negative.

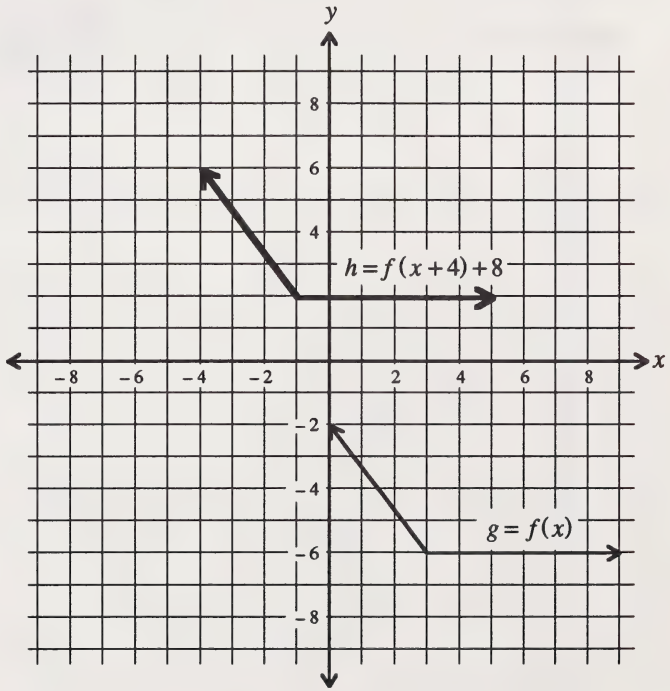
Example 5

The following is the graph of $g = f(x)$. Draw the graph of $h = f(x+4)+8$.



Solution:

The graph of h will be four units to the left and eight units upward since $a = -4$ and $b = +8$.



Example 6

- Write the new function produced when the graph of $y=\sqrt{3x}+1$ is moved five units down and six units to the left.

Solution:

Moving five units down means $b=-5$ and six units left means $a=-6$.

Therefore, the new function is as follows:

$$y=\sqrt{3(x-a)}+1+b$$
$$y=\sqrt{3(x-(-6))}+1-5$$
$$y=\sqrt{3(x+6)}-4$$

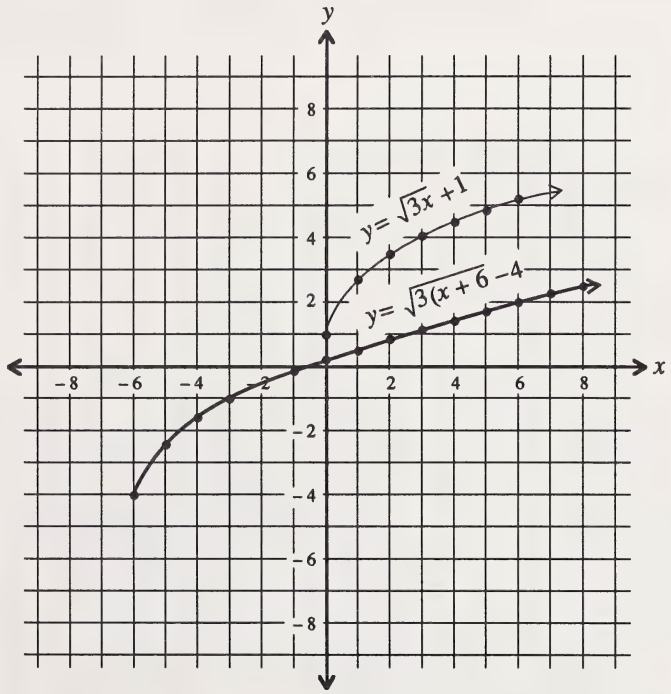
- Draw the graph of both functions.

Solution:

Make a table of values and plot the graph.

x	$y=\sqrt{3x}+1$	$y=\sqrt{3(x+6)}-4$
-6	undefined	$\sqrt{3(0)}-4=-4.0$
-5	undefined	$\sqrt{3(1)}-4\doteq-2.3$
-4	undefined	$\sqrt{6}-4\doteq-1.6$
-3	undefined	$\sqrt{9}-4=-1.0$
-2	undefined	$\sqrt{12}-4\doteq-0.5$
-1	undefined	$\sqrt{15}-4\doteq-0.1$
0	$\sqrt{0}+1=1.0$	$\sqrt{18}-4\doteq0.2$
1	$\sqrt{3}+1\doteq2.7$	$\sqrt{21}-4\doteq0.6$
2	$\sqrt{6}+1\doteq3.4$	$\sqrt{24}-4\doteq0.9$
3	$\sqrt{9}+1=4.0$	$\sqrt{27}-4\doteq1.2$
4	$\sqrt{12}+1\doteq4.5$	$\sqrt{30}-4\doteq1.5$
5	$\sqrt{15}+1\doteq4.9$	$\sqrt{33}-4\doteq1.7$

You can see that the graph is shifted six units to the left and five units down.

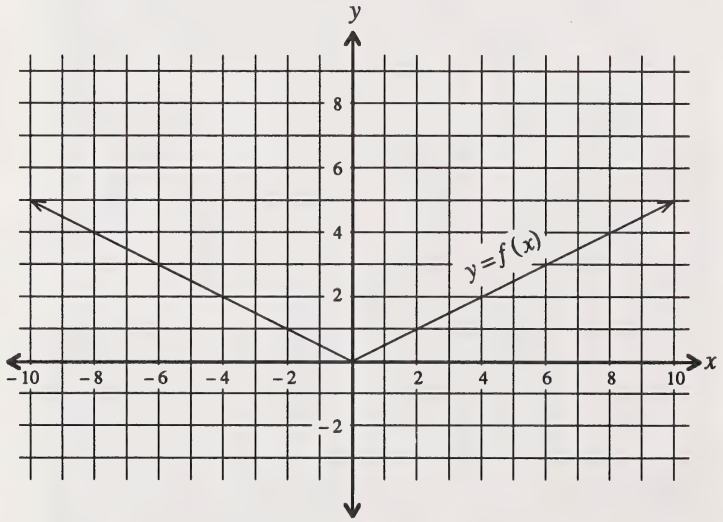


Do either a or b in each of the following questions. Graph paper is provided in **Appendix B.**

1. The following is the graph of $y = f(x)$.

   a. Draw the graph of $y = f(x)+5$.

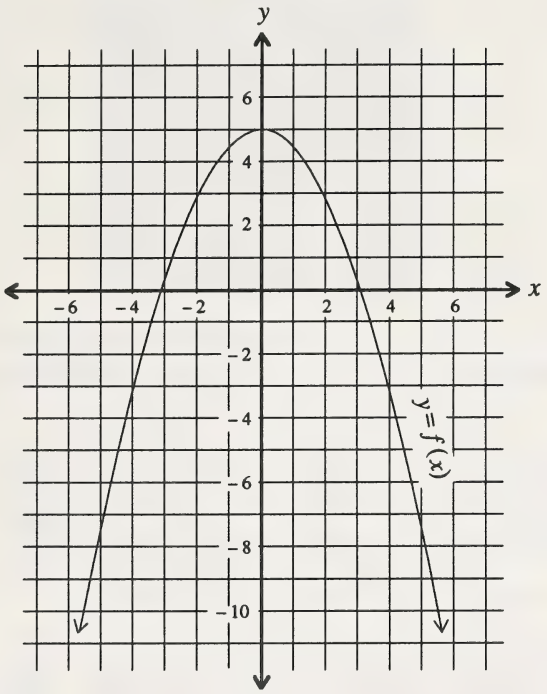
   b. Draw the graph of $y = f(x)-5$.



2. The following is the graph of $y = f(x)$.

   a. Draw the graph of $y = f(x-3)+4$.

   b. Draw the graph of $y = f(x+3)-4$.



3. Write the new function which describes the function $y = 2^x$ when it is moved as follows:

   a. three units vertically upwards

   b. five units vertically downwards

4. Write the new function which describes the function produced when $y = |x+5|$ is transformed by moving the graph as follows:

   a. three units to the left and four units upwards

   b. four units to the right and three units downwards

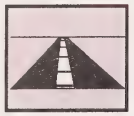
5. Write the new function which describes the function produced when $y = x^3 - 5$ is transformed by moving the graph as follows:

   a. six units to the right and five units downwards

   b. seven units to the left and two units upwards

For solutions to **Activity 1**, turn to **Appendix A, Topic 3**.

Activity 2

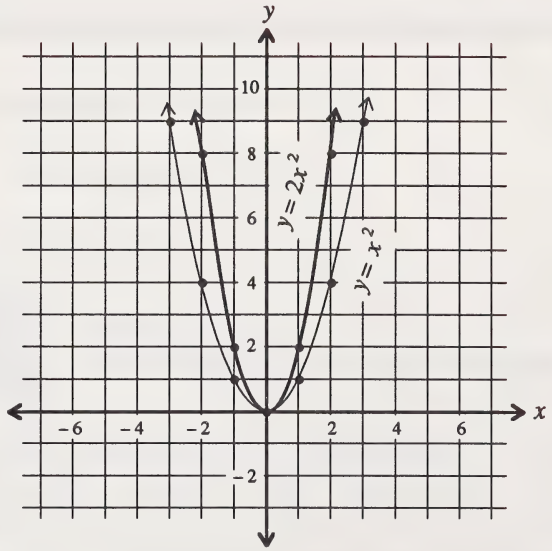


Investigate and describe the transformation effects of the parameter c in $y = cf(x)$ on the graph of $y = f(x)$.

To start this activity you may wish to view the program titled **Dilatation and Reflection of Functions.** This video is program 5 in the *Math Sense*[1] series and will give you a general picture of what dilatations and reflections of functions are about.

Examine the transformation of $y = x^2$ to $y = 2x^2$. The graphs of both functions are plotted as follows.

x	$y = x^2$	$y = 2x^2$
-3	9	18
-2	4	8
-1	1	2
0	0	0
1	1	2
2	4	8
3	9	18



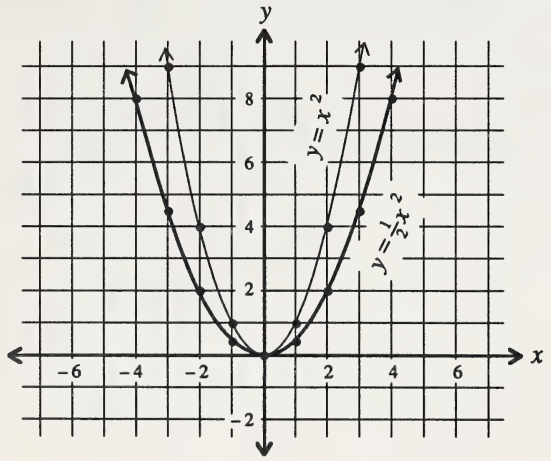
When you multiply x^2 by 2, you increase the y-values by a factor of 2. For example, when $x = 2$ in the graph of $y = x^2$, then $y = 4$. When $x = 2$ in the graph of $y = 2x^2$, then $y = 8$. The graph is stretched vertically by a factor of 2.

The transformation of $y = x^2$ to $y = 2x^2$ has the effect of making the graph narrower. The graph does not move vertically or horizontally as a whole. Its shape changes.

Examine the transformation of $y = x^2$ to $y = \frac{1}{2}x^2$. What do you predict will happen?

[1] *Math Sense* is a title of ACCESS Network.

x	$y=x^2$	$y=\frac{1}{2}x^2$
−3	9	4.5
−2	4	2.0
−1	1	0.5
0	0	0.0
1	1	0.5
2	4	2.0
3	9	4.5
4	16	8.0

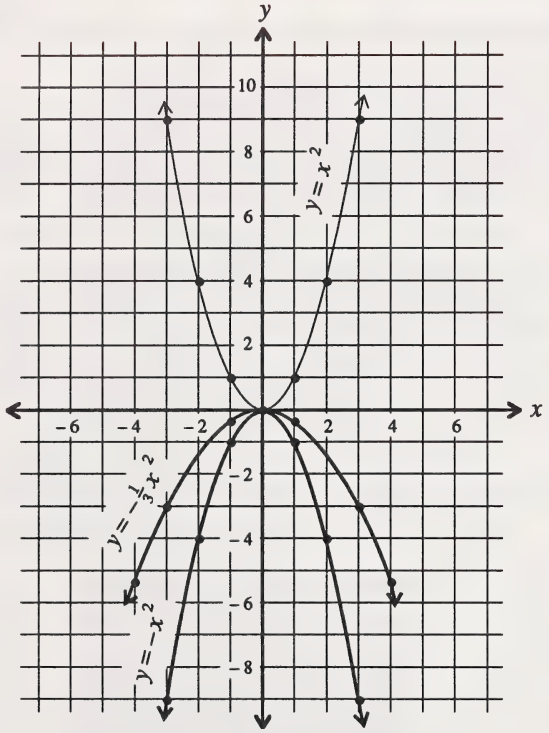


When you multiply x^2 by $\frac{1}{2}$, you decrease the y-values by a factor of $\frac{1}{2}$. For example, when $x=2$ in the graph of $y=x^2$, then $y=4$. When $x=2$ in the graph of $y=\frac{1}{2}x^2$, then $y=2$. The graph is compressed vertically by a factor of $\frac{1}{2}$ of the original.

The transformation of $y=x^2$ to $y=\frac{1}{2}x^2$ has the effect of making the graph wider. The graph does not move horizontally or vertically. The shape of the graph changes.

What will the effect be if c is negative? Examine the transformations of $y=x^2$ to $y=-x^2$ and to $y=-\frac{1}{3}x^2$.

x	$y=x^2$	$y=-x^2$	$y=-\frac{1}{3}x^2$
−3	9	−9	−3
−2	4	−4	$\frac{-4}{3}$
−1	1	−1	$\frac{-1}{3}$
0	0	0	0
1	1	−1	$\frac{-1}{3}$
2	4	−4	$\frac{-4}{3}$
3	9	−9	−3



The graph of $y = -x^2$ is the mirror image of $y = x^2$ in the line $y = 0$. The graph of $y = -\frac{1}{3}x^2$ is wider than the graph of $y = -x^2$.

In general the transformation effects of the parameter c in $y = cf(x)$ on the graph of $y = f(x)$ are as follows:

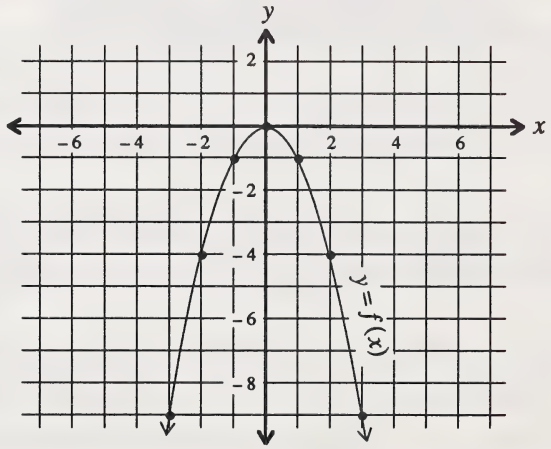
- If $0 < c < 1$, the graph becomes wider.
- If $c > 1$, the graph becomes narrower.
- If $-1 < c < 0$, the graph is wider and is reflected in the x-axis.
- If $c < -1$, the graph is narrower and is reflected in the x-axis.

Do either a or b in each question. Graph paper is provided in **Appendix B.**

1. The graph of $y = f(x)$ is drawn on the following grid.

   a. Draw the graph of $y = -2f(x)$.

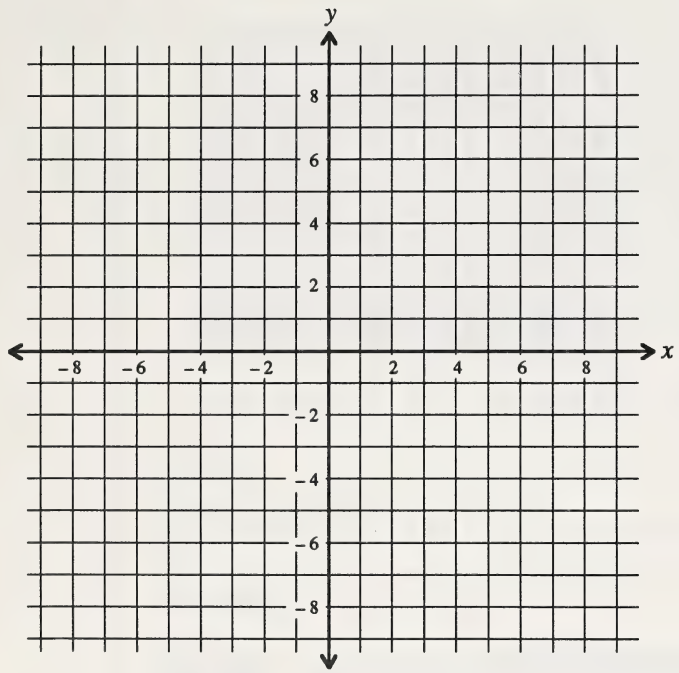
   b. Draw the graph of $y = \frac{1}{2}f(x)$.



2. Draw the graph of $y = |x|$ and the graph of the following:

   a. $y = |x + 3|$

   b. $y = -\frac{1}{3}|x + 3|$



3. Write the new function which describes the transformation of the following:

   a. $y = 2^x$ compressed vertically by a factor of $\frac{1}{3}$ of the original

   b. $y = 2^x$ stretched vertically by a factor of 3

4. Write the new function which describes the transformation of the following:

   a. $g(x) = 2^{3x}$ reflected in the x-axis

   b. $g(x) = 2^{3x}$ compressed vertically by a factor of $\frac{1}{2}$ of the original and reflected in the x-axis

For solutions to **Activity 2**, turn to **Appendix A, Topic 3**.

Activity 3

Describe and draw the graphs of
$y = cf(x-a)+b$.

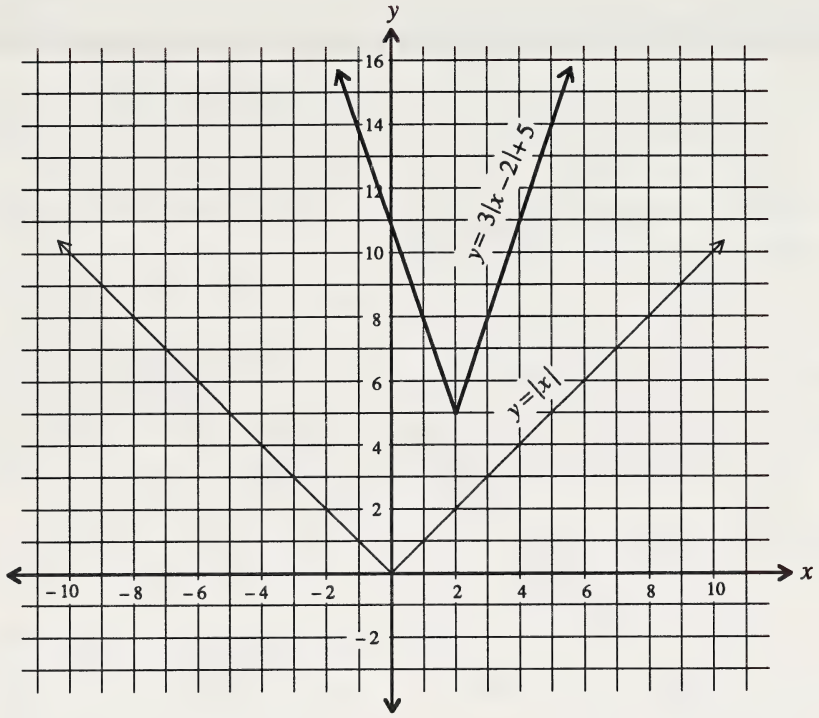
Example 7

Draw the graphs of $y = |x|$ and $y = 3|x-2|+5$ and describe the graphs.

Solution:

Make a table of values and draw the graph.

x	$y=\|x\|$	$y=3\|x-2\|+5$
10	10	29
5	5	14
3	3	8
2	2	5
1	1	8
0	0	11
−1	1	14
−5	5	26
−10	10	41



The graphs can be described as follows.

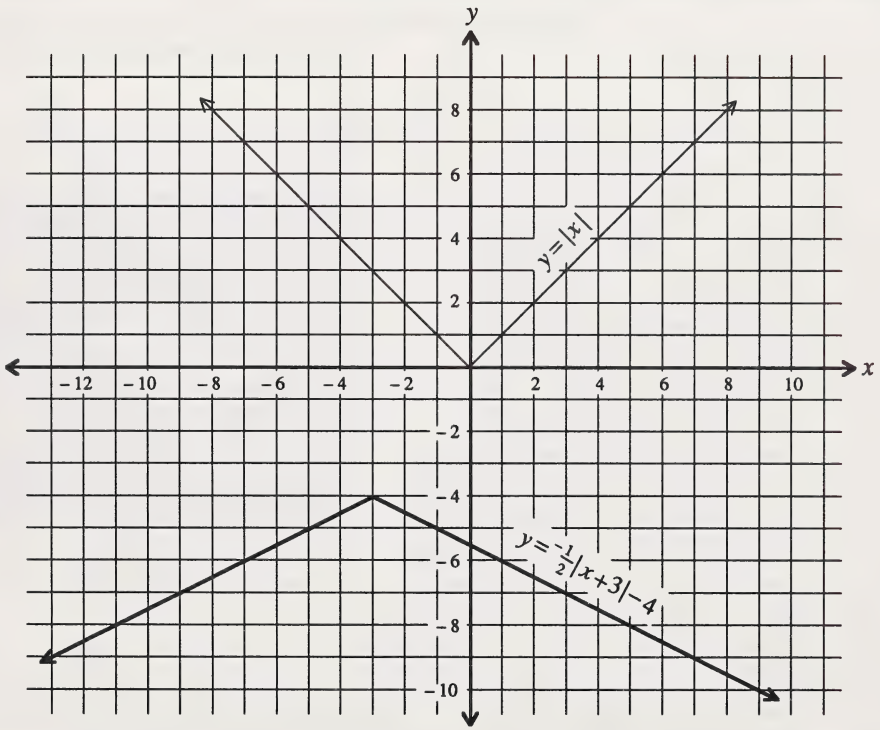
- If $c = 3$, the graph of $y = 3|x-2|+5$ is a narrower version of $y = |x|$. It has been stretched vertically by a factor of 3.
- If $a = 2$, the graph of $y = 3|x-2|+5$ is two units to the right of $y = |x|$.
- If $b = 5$, the graph of $y = 3|x-2|+5$ is five units upwards from $y = |x|$.

Example 8

Draw the graphs of $y = |x|$ and $y = -\frac{1}{2}|x+3| - 4$ and describe the graphs.

Solution:

The graphs are shown as follows.



The graphs can be described as follows.

- If $c = -\frac{1}{2}$, the graph of $y = |x|$ is compressed vertically by a factor of $\frac{1}{2}$ of the original and reflected in the x-axis.
- If $a = -3$, the graph of $y = |x|$ is moved three units to the left.
- If $b = -4$, the graph of $y = |x|$ is moved four units down.

In general the descriptions of $y = cf(x-a)+b$ compared to $y = f(x)$ are as follows.

- If $c < 0$, the graph of $y = f(x)$ is reflected in the x-axis.
- If $c < -1$ or $c > 1$, the graph of $y = f(x)$ is stretched vertically by a factor of c or made narrower.
- If $-1 < c < 1$, the graph of $y = f(x)$ is compressed vertically by a factor of c or made wider.
- If $a > 0$, the graph of $y = f(x)$ moves a units to the right.
- If $a < 0$, the graph of $y = f(x)$ moves a units to the left.
- If $b > 0$, the graph of $y = f(x)$ moves b units upwards.
- If $b < 0$, the graph of $y = f(x)$ moves b units downwards.

Do either a or b in each question. Graph paper is provided in **Appendix B.**

1. Write the new function which describes the transformation of the following:

   a. $y = \sqrt{x}$ compressed vertically by a factor of $\frac{1}{2}$ of the original, moved to the right five units, and moved downward three units

   b. $y = \sqrt{x}$ stretched vertically by a factor of 2, moved to the left five units, and moved upward three units

2. Write the new function which describes the transformation of the following:

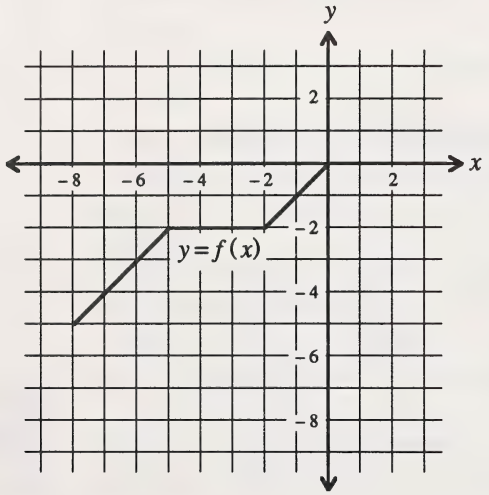
   a. $y = |x|$ moved to the left three units, moved downward seven units, reflected in the x-axis, and compressed vertically by a factor of $\frac{1}{3}$ of the original

   b. $y = |x|$ moved to the right three units, moved upward seven units, and stretched vertically by a factor of 3

3. The graph of $y = f(x)$ is shown. Draw the graph of the following:

   a. $y = -f(x)$

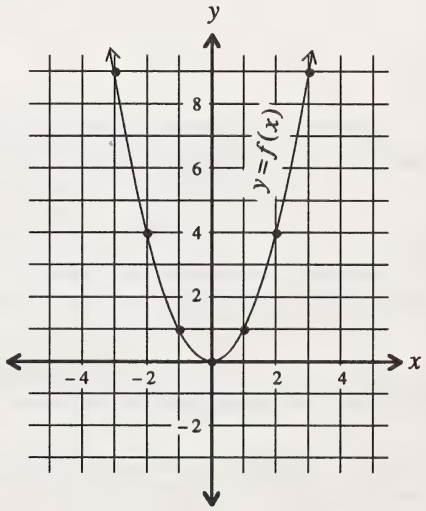
   b. $y = f(x-9)+5$



4. The graph of $y = f(x)$ is shown. Draw the graph of the following:

   a. $y = -\frac{1}{2}f(x+6)+4$

   b. $y = 2f(x-3)-4$



For solutions to **Activity 3**, turn to **Appendix A, Topic 3**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

The Extra Help in this topic consists of three parts. Each part is a summary of this topic. **Part A** gives this summary through the print mode. **Part B** consists of a software program which can be used for extra work on this topic. **Part C** consists of a video program which summarizes the topic. You may do any part or all three parts if you wish.

Part A

The transformations discussed can be summarized into one equation.
The basic function which is transformed is $y = f(x)$.

The transformation is $y = cf(x-a)+b$.

Effect on $y = f(x)$

- reflects $f(x)$ in the x-axis when $c < 0$, that is, when c is negative
- makes $f(x)$ wider when $-1 < c < 1$
- makes $f(x)$ narrower when $c < -1$ or $c > 1$

Effect on $y = f(x)$

- moves $f(x)$ to the right when $a > 0$
- moves $f(x)$ to the left when $a < 0$

Effect on $y = f(x)$

- moves $f(x)$ up when $b > 0$
- moves $f(x)$ down when $b < 0$

Use the previous descriptions to do the following questions.

Compare the transformed function to the basic function in each part and explain clearly what transformations occurred. Graph paper is provided in **Appendix B.**

1. $y=\frac{1}{x}$ $y=\frac{1}{x}+5$

2. $y=(x-2)^2$ $y=3(x-2)^2$

3. $y=x^2$ $y=x^2-5$

4. $y=x^2$ $y=-(x+5)^2+3$

5. $y=x^2$ $y=2(x-3)^2-7$

6. $y=x^2$ $y=-\frac{1}{3}(x+4)^2-6$

For solutions to **Extra Help**, turn to **Appendix A, Topic 3.**

Part B

Use the computer program titled *Graphing Equations*[1], Part 1 – Equation Plotter to investigate the effects of the three parameters (*a*, *b*, and *c*) discussed in this topic. Use the program to compare two or three graphs at once. For example, enter the equation $y=x$ and plot its graph. Then enter and plot $y=x+2$ and $y=3x+2$. All three graphs will appear on the screen. Compare the effects each change has on the equation.

Part C

If you have access to a videocassette recorder (VCR), you may wish to view the video titled **Transformation of Functions.** This video is program 6 in the *Math Sense*[2] series.

[1] *Graphing Equations* is a title of Conduit.
[2] *Math Sense* is a title of ACCESS Network.

Extensions

This topic covered functions up to quadratic equations. What about other functions? Do the same transformation effects apply? Investigate transformations of some other functions.

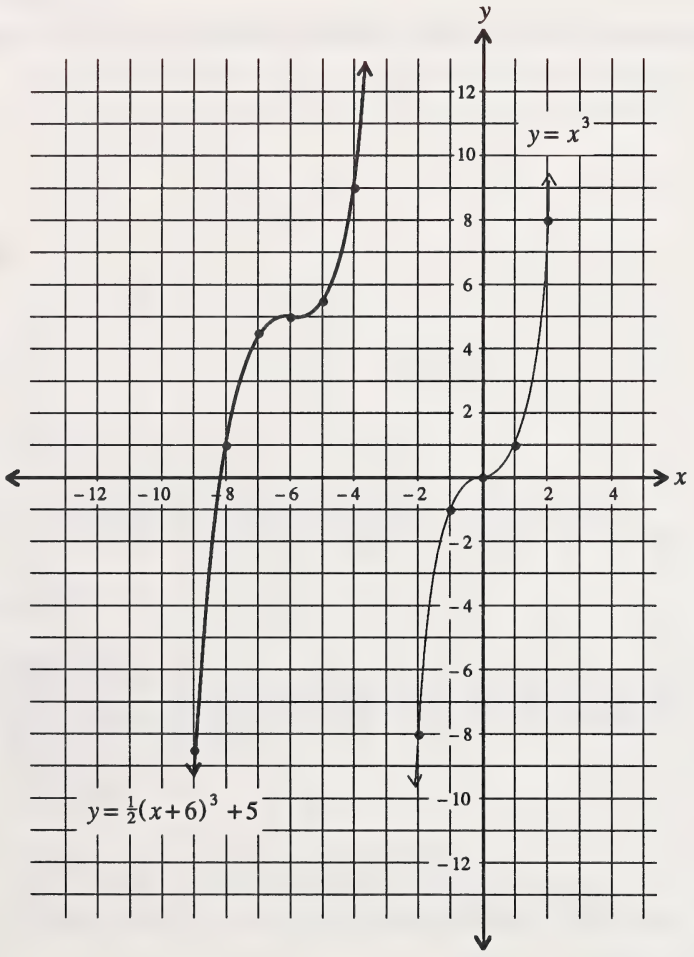
Example 9

Does the transformation of $y = x^3$ to $y = \frac{1}{2}(x+6)^3 + 5$ have the same predictable effects according to the generalizations you learned in this topic?

Solution:

Make a table of values for each function. Several values in the table have been left blank because they are large numbers.

x	$y = x^3$	$y = \frac{1}{2}(x+6)^3 + 5$
−9		−8.5
−8		1.0
−7		4.5
−6		5.0
−5	−125	5.5
−4	−64	9.0
−3	−27	18.5
−2	−8	
−1	−1	
0	0	
1	1	
2	8	
3	27	
4	64	
5	125	



Look at the graphs of $y = x^3$ and its transformation $y = \frac{1}{2}(x+6)^3 + 5$.

The point $(0,0)$ has been moved to $(-6,5)$ which corresponds to the values of $a = -6$ and $b = 5$ moving the graph six units to the left and five units up.

The graph of $y = \frac{1}{2}(x+6)^3 + 5$ is wider than that of $y = x^3$ which is consistent with the general statement that when $-1 < c < 1$, the graph is wider.

Example 10

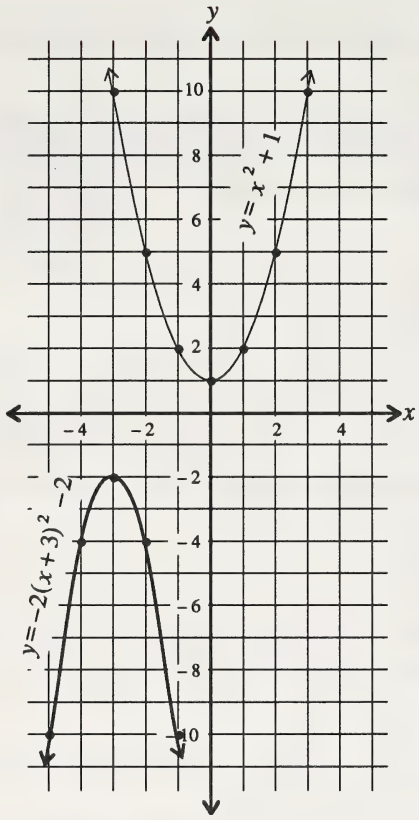
Does the transformation of $y = x^2 + 1$ to $y = -2(x+3)^2 - 2$ have the same predictable effects according to the generalizations of $y = cf(x-a)+b$?

Solution:

The function $y = x^2 + 1$ is transformed to $y = -2(x+3)^2 - 2$.

Make a table of values and plot both graphs.

x	$y = x^2 + 1$	$y = -2(x+3)^2 - 2$
−6	37	−20
−5	26	−10
−4	17	−4
−3	10	−2
−2	5	−4
−1	2	−10
0	1	−20
1	2	−34
2	5	
3	10	



The effects of the transformation are as follows:

- Since $c = -2$, the graph of $y = x^2 + 1$ is inverted (reflected in the x-axis) and made narrower.
- Since $a = -3$, the graph of $y = x^2 + 1$ is moved three units to the left.
- Since $b = -2$, the graph of $y = x^2 + 1$ is **not moved two units down** but rather is moved three units down. (This value went from +1 to −2 which is three units.)

What happened?

Did you notice that $y = x^2 + 1$ does not cross the x-axis?

You can give many examples of this idea. In general if the graph of $y = f(x)$ does not intersect the x-axis, then the transformation constant c may have an effect on the vertical translation.

The questions in this course will intersect the x-axis so that you will be able to make the correct predictions from the generalization $y = cf(x-a)+b$.

Try the following questions. Graph paper is provided in **Appendix B.**

1. Will the transformation of $y = 2^x$ to $y = \left(2^{x+6}\right)+4$ have the following characteristics? Check your answer by graphing both functions.

   a. Is $y = \left(2^{x+6}\right)+4$ narrower than $y = 2^x$?

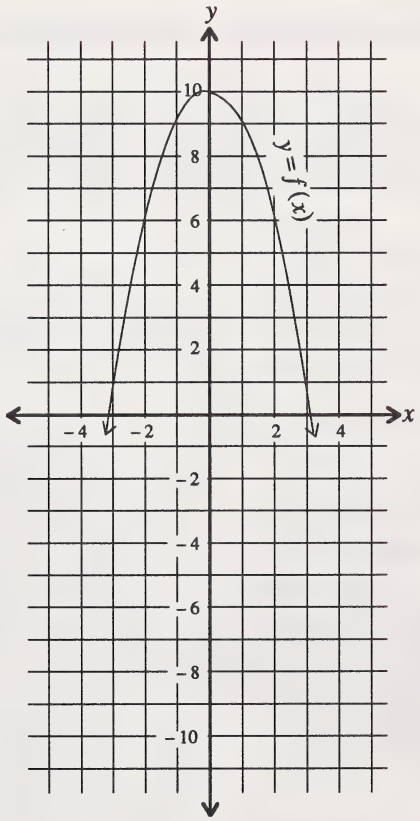
   b. Is $y = \left(2^{x+6}\right)+4$ six units to the left of $y = 2^x$?

   c. Is $y = \left(2^{x+6}\right)+4$ four units higher than $y = 2^x$?

2. Given the basic function $y = f(x)$ as drawn, draw the graph of the following:

   a. $y = \frac{1}{2}f(x-2)+3$

   b. $y = -f(x+1)-5$



For solutions to **Extensions**, turn to **Appendix A, Topic 3.**

Unit Summary

What You Have Learned

In Topic 1 you studied graphing of relations. You learned to

- graph relations that describe everyday occurrences and physical phenomena
- solve problems by graphing and interpreting the graphs which describe everyday occurrences and physical phenomena

In Topic 2 functions were studied. You learned to

- represent simple mathematical situations such as direct and inverse variations with tables of values, identify the dependent and independent variables, and express the domain and the range noting any restrictions
- write functions using functional notation and evaluate them by substitution for the domain
- graphically and algebraically illustrate and define different kinds of functions, in particular, polynomial (linear, quadratic, and cubic), reciprocal, absolute-value, square-root, and exponential functions

Unit Summary

- solve problems involving functions algebraically and by drawing, interpolating, extrapolating, interpreting, and analyzing graphs of functions

In Topic 3 transformations were studied. You learned to

- investigate and describe the transformation effects of the parameters a and b in $y = f(x-a)+b$ on the graph of $y = f(x)$
- investigate and describe the transformation effects of the parameter c in $y = cf(x)$ on the graph of $y = f(x)$
- describe and draw the graphs of $y = cf(x-a)+b$

You are now ready to complete the **Unit Assignment**.

Appendices

Appendix A
Solutions

Review

Topic 1 Graphing Relations

Topic 2 Functions

Topic 3 Transformations of Functions

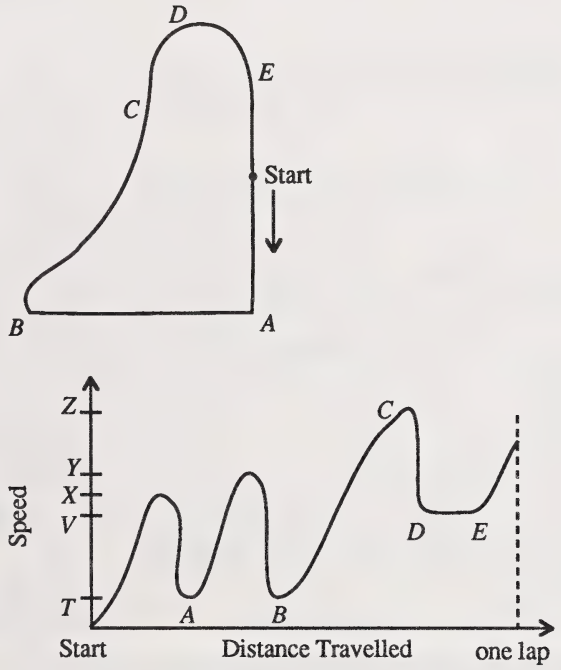
Appendix B
Graphing Material

Graph Paper

Appendix A
Solutions

Review

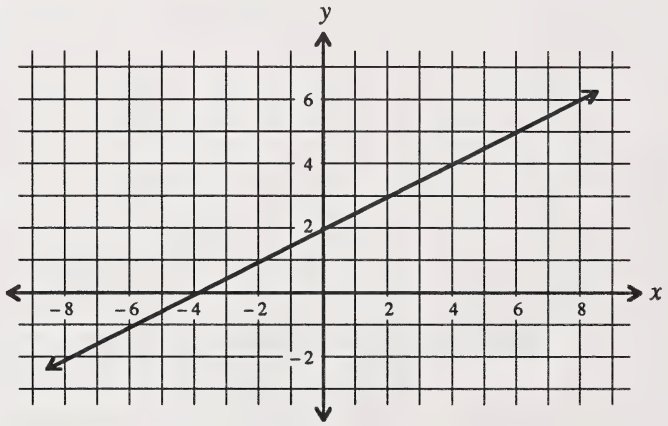
1.



You accelerate from a standing start and reach a certain speed X, then you have to brake hard for the right-angled corner at A. You can safely negotiate the corner at speed T. You then accelerate towards B until you reach a speed Y and have to brake hard for the right-angled corner at B. You safely negotiate corner B at speed T and then accelerate hard towards C, slowing your acceleration around curve C, and slowing down towards D. Before D, you brake hard to reach the safe cornering speed V which you maintain around the large corner. Finally, you finish the corner and accelerate hard towards the start.

2. The graph of $y = \frac{1}{2}x + 2$ is as follows:

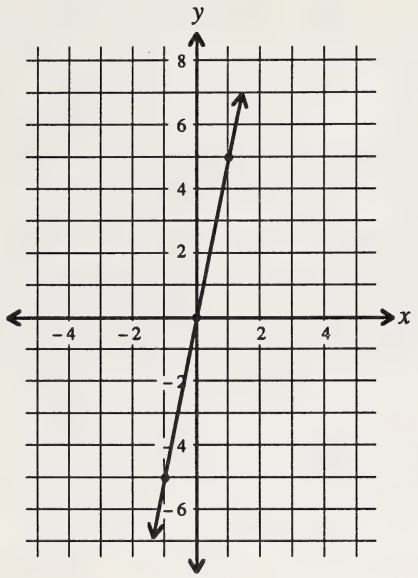
x	-4	0	4
y	0	2	4



This would represent a partial variation since the equation can be rewritten as $y = 2 + \frac{1}{2}x$.

3. The graph of $y = 5x$ is as follows:

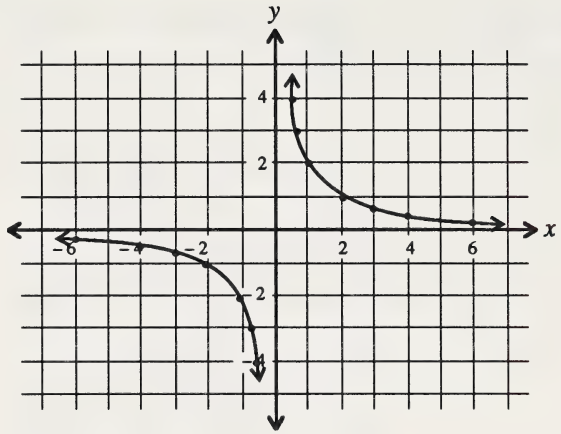
x	-1	0	1
y	-5	0	5



This represents a direct variation.

4. Rewrite $xy = 2$ as $y = \frac{2}{x}$ and set up a table of values.

x	-6	-4	-3	-2	-1	$\frac{1}{2}$	1	2	3	4	6
y	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	-1	-2	4	2	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$

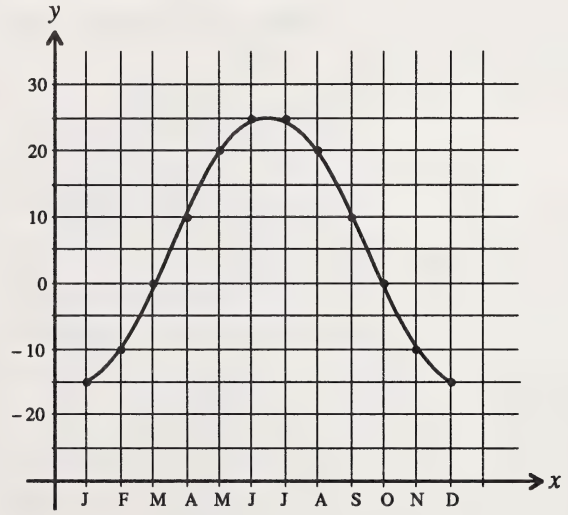


This is an inverse relation.

5. Graph C is the answer since it closely represents Edmonton's cold winter months. Warming takes place in March, April, and May. It becomes hotter in June, July, and August and cools down again during September, October, and November.

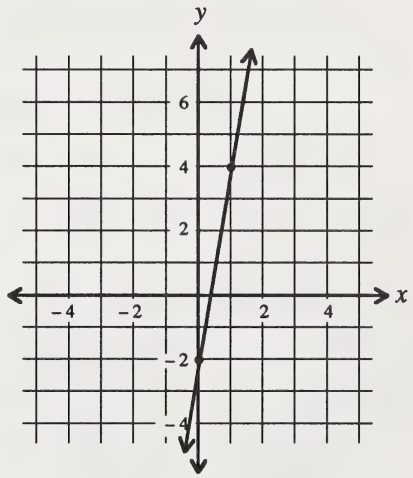
   Graph A indicates a hot winter and a cold summer.
   Graph B indicates the same temperature year-round.
   Graph D indicates sharp temperature variation with no warm summer months.

C.



6. If the y-intercept is equal to -2, then the coordinates of the point are $(0, -2)$.

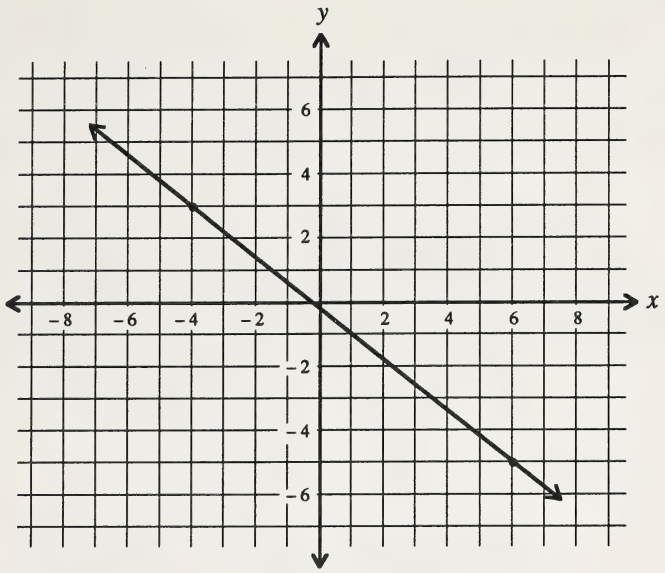
   Plot the point $(0, -2)$. From this point, since the slope is 6, move up six units and to the right one unit. Plot this point and draw the line by joining the points.



The reason you went up six units and to the right one unit is to get a slope of 6. Slope means $\frac{\textbf{rise}}{\textbf{run}}$ so you can select any ratio that will produce the required slope of 6.

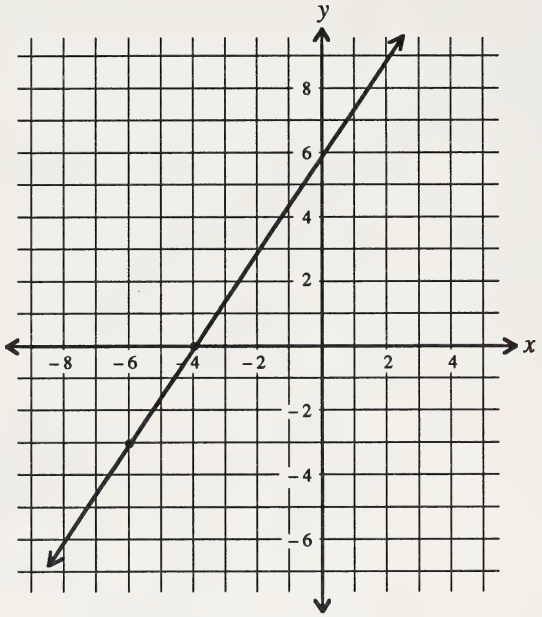
For example, $\frac{12}{2}$, $\frac{18}{3}$, $\frac{24}{4}$ . . .

7. Plot the points $(-4, 3)$ and $(6, -5)$, then join the points with a straight line.



8. The slope of the line $y = -\frac{2}{3}x - 9$ is $-\frac{2}{3}$. The slope of the line perpendicular to the line $y = -\frac{2}{3}x - 9$ is $\frac{3}{2}$. (Take the negative reciprocal of $-\frac{2}{3}$).

Plot the point $(-6, -3)$. From this point, since the slope is $\frac{3}{2}$, go up three units and to the right two units. Plot this point and join the points with a straight line.

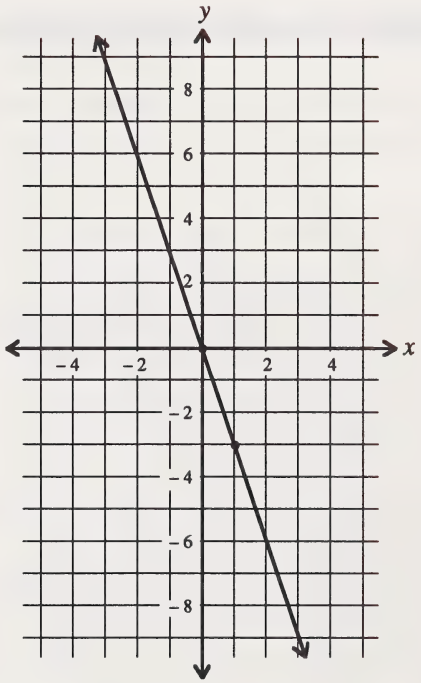


9. Solve the equation for y and put it into slope-intercept form.

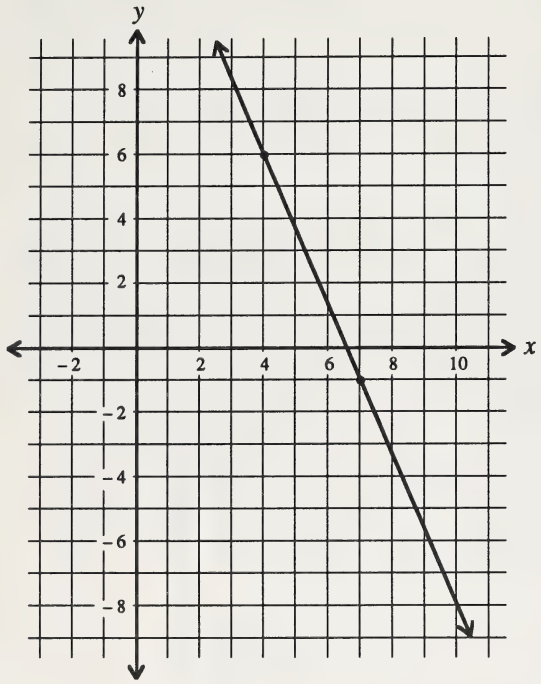
$$6x+2y=5$$
$$2y=-6x+5$$
$$y=-3x+\frac{5}{2}$$

The slope of the line is -3. The slope of any line parallel to this line is -3.

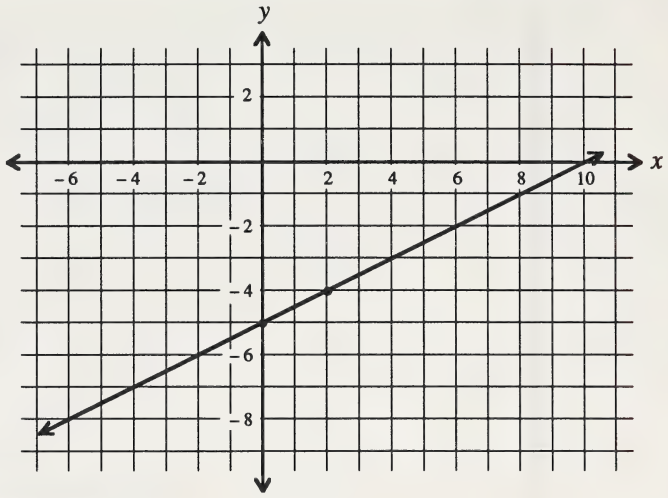
Plot the point $(0,0)$. From this point, go down three units and to the right one unit. Plot this point and join the points with a straight line.



10. Plot the point $(4, 6)$. From this point, go down seven units and to the right three units. Join these points with a straight line.



11. The slope of the line $y = \frac{1}{2}x - 3$ is $\frac{1}{2}$. The slope of any line parallel to this line is $\frac{1}{2}$. The y-intercept is equal to -5 and is the point $(0, -5)$. Plot this point and from it go up one unit and to the right two units. Join these points with a straight line.

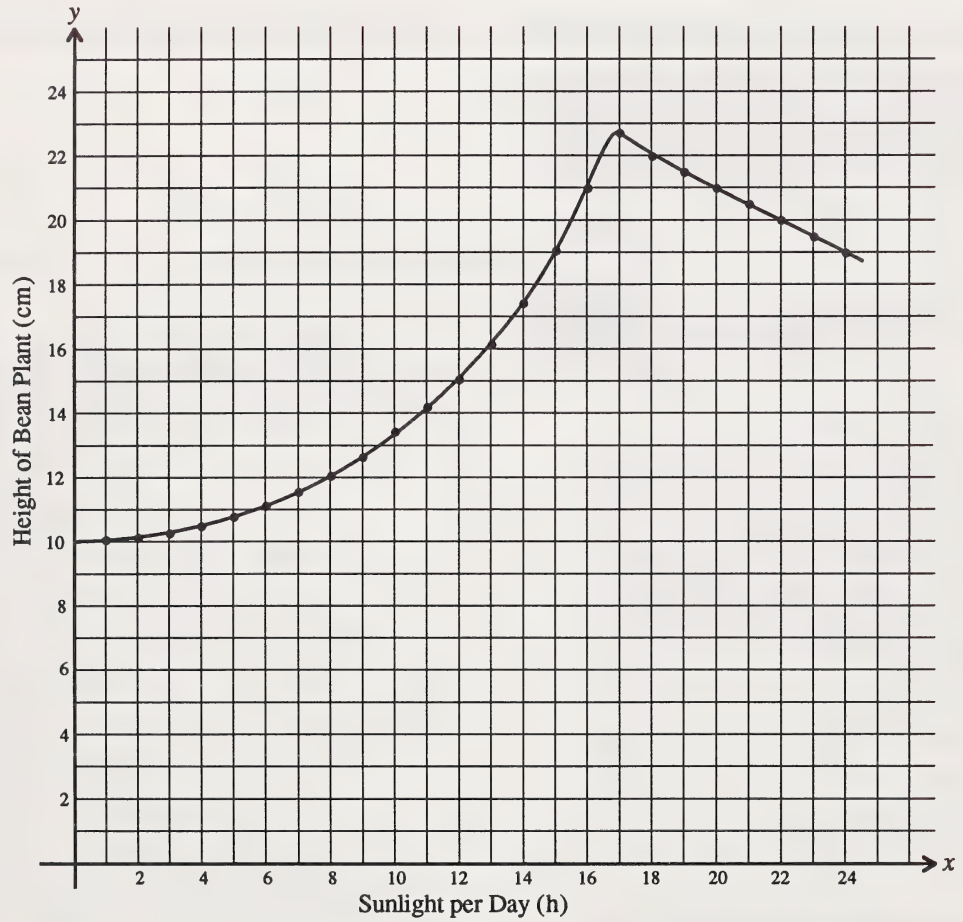


Exploring Topic 1

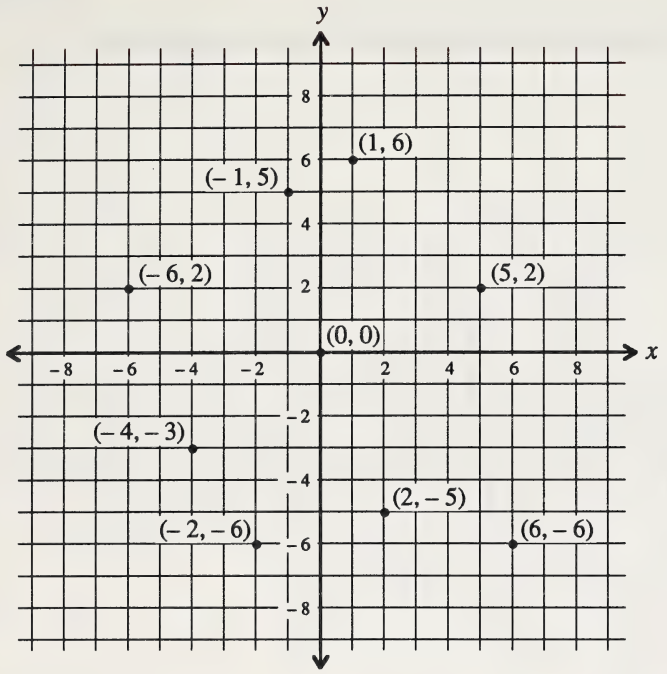
Activity 1

Graph relations that describe everyday occurrences and physical phenomena.

1.

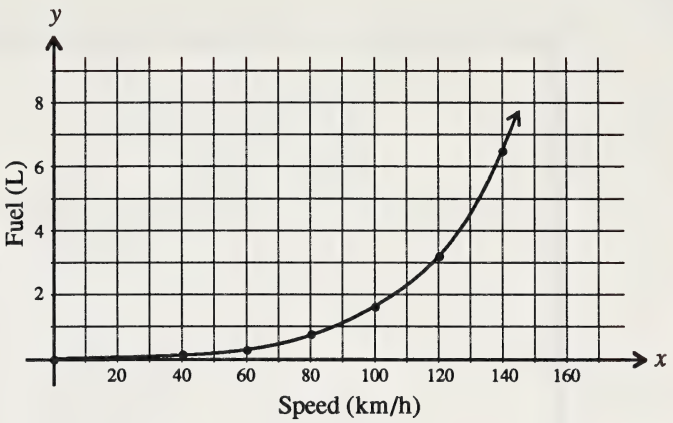

y
24
22
20
18
16
14
12
10
8
6
4
2
Height of Bean Plant (cm)
2
4
6
8
10
12
14
16
18
20
22
24
x
Sunlight per Day (h)

2.



3.

Speed (km)	Fuel consumed (L)
0	0.0
40	0.2
60	0.4
80	0.8
100	1.6
120	3.2
140	6.4



4. Make a table of values, then plot the points.

$y = -x^2 + 8$

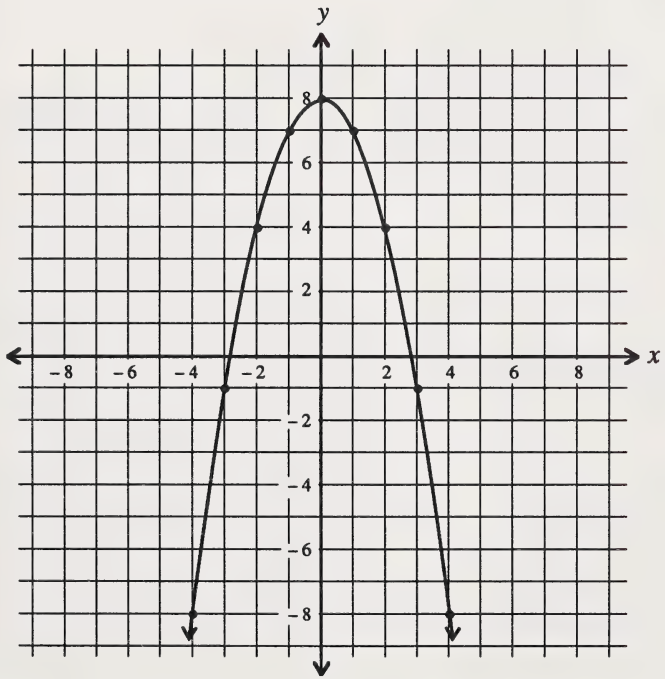
x	y
– 4	– 8
– 3	– 1
– 2	4
– 1	7
0	8
1	7
2	4
3	– 1
4	– 8

The following are sample calculations.

$x=-4,\ y=-(-4)^2+8$
$=-16+8$
$=-8$

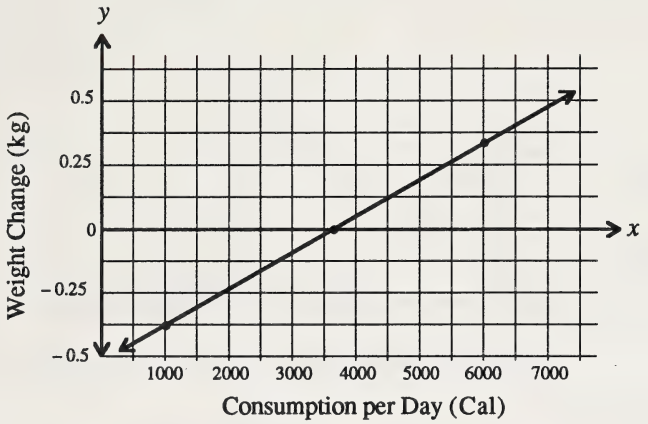
$x=-3,\ y=-(-3)^2+8$
$=-9+8$
$=-1$

$x=3,\ y=-(3)^2+8$
$=-9+8$
$=-1$



5.

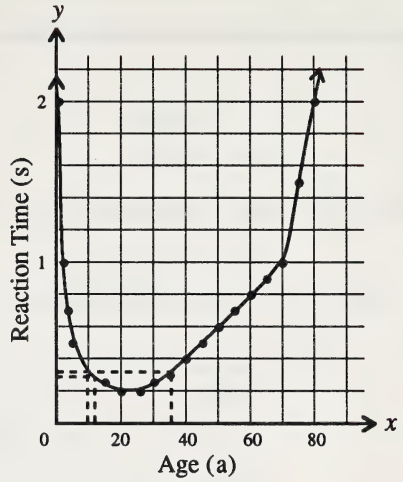
x	y
1000	−0.375
3600	0
6000	0.329



Activity 2

Solve problems by graphing and interpreting the graphs which describe everyday occurrences and physical phenomena.

1.

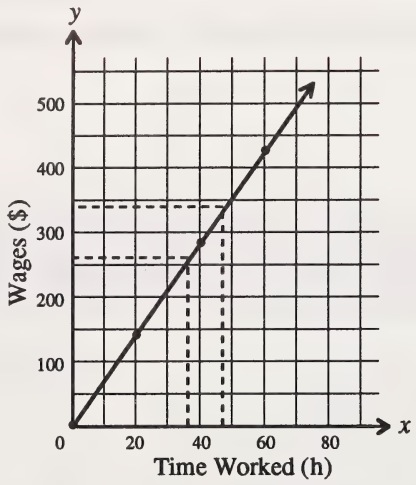


a. According to the graph the reaction time of a twelve-year-old is 0.30 s.

b. According to the graph a person with a reaction time of 0.35 s would be ten years old or thirty-eight years old.

2. Make a table of values, then plot the graph.

Time worked (h)	Wages ($)
0	0.00
1	7.25
10	72.50
20	145.00
60	435.00



a. According to the graph, she would have to work 48 h.

b. According to the graph, she would earn about $270.

3.

Time (s)	Height (m)
0	0.0
1	95.1
2	180.4
3	255.9
4	321.6
5	377.5
6	423.6
7	459.9
8	486.4
9	503.1
10	510.0
11	507.1
12	494.4
13	471.9
14	439.6
15	397.5
16	345.6
17	283.9
18	212.6
19	131.1
20	40.0
21	– 60.9

The following are sample calculations.

$$\text{If } t = 1 \text{ s, } h = (100 \text{ m/s})(1 \text{ s}) + \frac{1}{2}\left(-9.8 \text{ m/s}^2\right)(1 \text{ s})^2$$

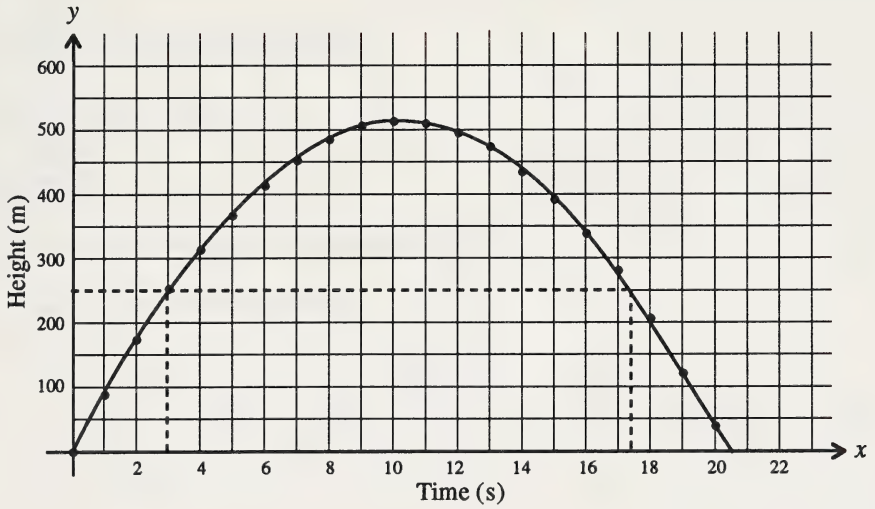
$$= 100 \text{ m} - 4.9 \text{ m}$$

$$= 95.1 \text{ m}$$

If $t = 2$ s, $h = (100 \text{ m/s})(2 \text{ s}) + \frac{1}{2}\left(-9.8 \text{ m/s}^2\right)(2 \text{ s})^2$

$= 200 \text{ m} - 19.6 \text{ m}$

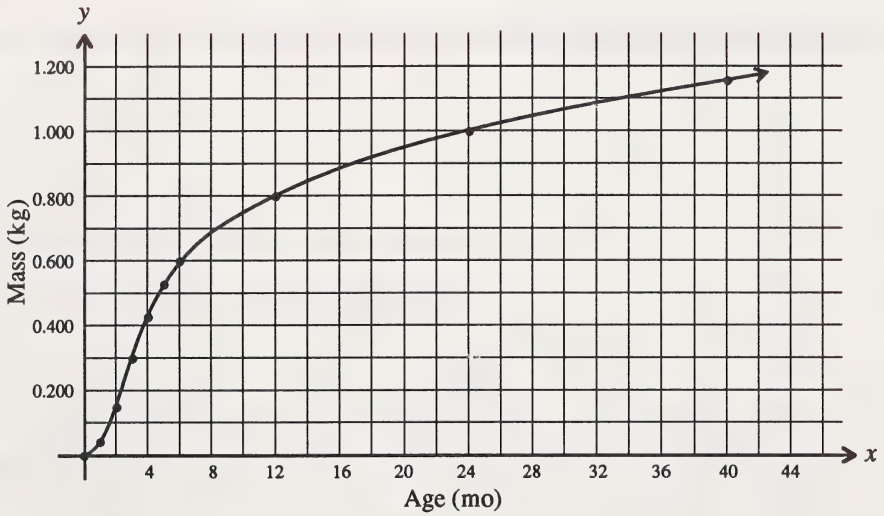
$= 180.4 \text{ m}$



a. The flare will travel approximately 510 m.

b. The height will be 250 m at approximately 3 s and 17.5 s.

4.



a. The fish is seventeen months old.

b. The three-year-old fish has a mass of 1.13 kg.

Extra Help

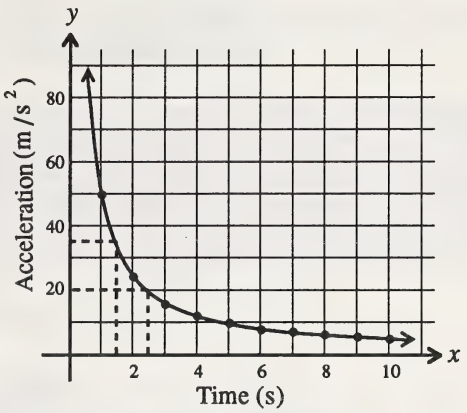
1. Make a table of values.

The following is a sample calculation.

$$\text{If } t = 5 \text{ s}, \ a = \frac{50 \text{ m/s}}{5 \text{ s}}$$

$$= 10.00 \text{ m/s}^2$$

Time (s)	Acceleration $\left(m/s^2\right)$
0	not possible
1	50.00
2	25.00
3	16.67
4	12.50
5	10.00
6	8.33
7	7.14
8	6.25
9	5.56
10	5.00

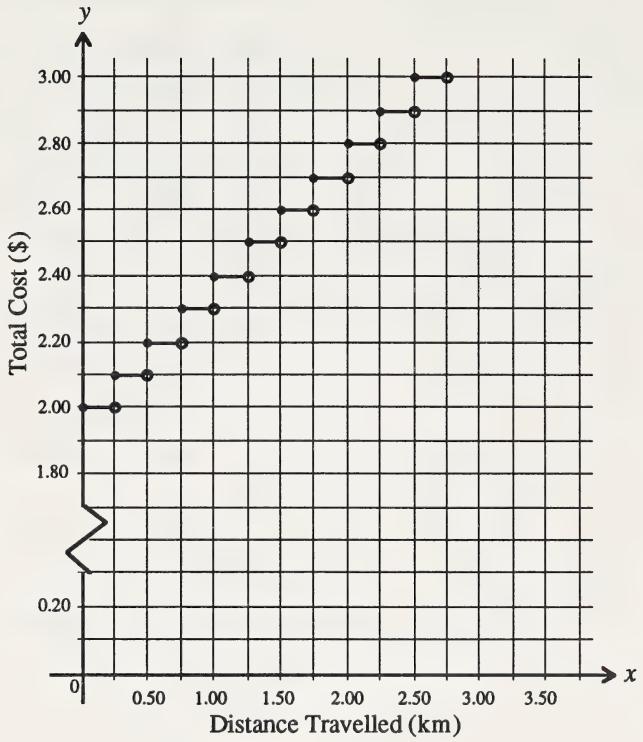


2. a. According to the graph the acceleration would be about $35\ m/s^2$ when the time is 1.5 s.

b. According to the graph the time is 2.5 s when the acceleration is $20\ m/s^2$.

Extensions

The graph of the taxi fare step function is drawn as follows.



This symbol indicates a break from the ordered increments in the scale. The graph would be much too large if all values of the total cost were included from \$0.00 to \$3.00.

Exploring Topic 2

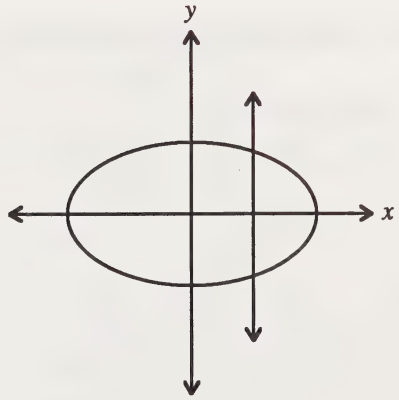
Activity 1

Represent simple mathematical situations such as direct and inverse variations with tables of values, identify the dependent and independent variables, and express the domain and the range noting any restrictions.

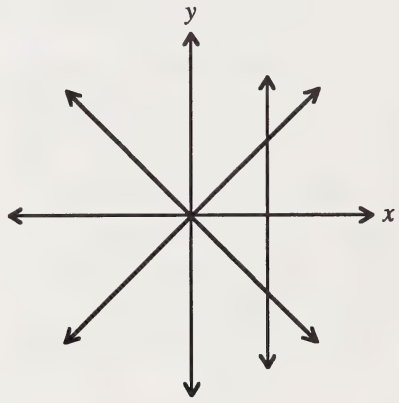
1. Both b and c are functions.
   Answer a has two points with the same x-coordinates.
   They are $(0, -7)$ and $(0, 0)$.

2. Both c and d are functions.
   The vertical-line test fails in a and b as shown.

a.



b.



3. Answers a, c, e, f, g, and h represent functions.
Answer b is not a function.

$$y^2 - x = 0$$
$$y^2 = x$$
$$y = \pm\sqrt{x}$$

Therefore, there are two values for the square root of x.

Answer d is not a function.

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}$$

Answer h is a function since $\sqrt{x}$ means the principal square root of x which is the positive square root.

4. a. The dependent variable is y.
The independent variable is x.

b. The dependent variable is y.
The independent variable is x.

c. The dependent variable is y.
The independent variable is x.

d. The dependent variable is y.
The independent variable is x.

e. The dependent variable is d.
The independent variable is t.

Time is usually the independent variable; other variables usually depend on time.

f. The dependent variable is y.
The independent variable is x.

g. The dependent variable is y.
The independent variable is x.

h. The dependent variable is y.
The independent variable is x.

5. a. Domain: R (R is the set of real numbers.)
Range: R

b. Domain: $x \geq 0$, $x \in R$
Range: R

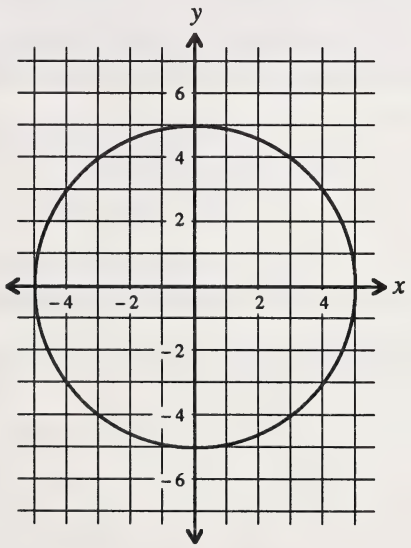
c. Domain: R
Range: R

d. Domain: $-5 \le x \le 5,\ x \in R$
Range: $-5 \le y \le 5,\ y \in R$

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}$$

Since you cannot take the square root of a negative number, $25 - x^2$ must always be greater than or equal to 0, that is, $25 - x^2 \ge 0$. Therefore, $x^2 \le 25$, where $-5 \le x \le 5$. The same logic applies to y.

The graph of $x^2 + y^2 = 25$ is a circle.



e. Domain: $t \ge 0,\ t \in R$
Range: $d \le 0,\ d \in R$
Since acceleration (a) is negative, d will always be negative in $d = \frac{1}{2}at^2$.

f. Domain: $x \in R$
Range: $y \ge 0,\ y \in R$

g. Domain: $x \in R,\ x \ne 0$
Range: $y > 0,\ y \in R$

When x is negative, x^2 is positive; hence, y is always positive.

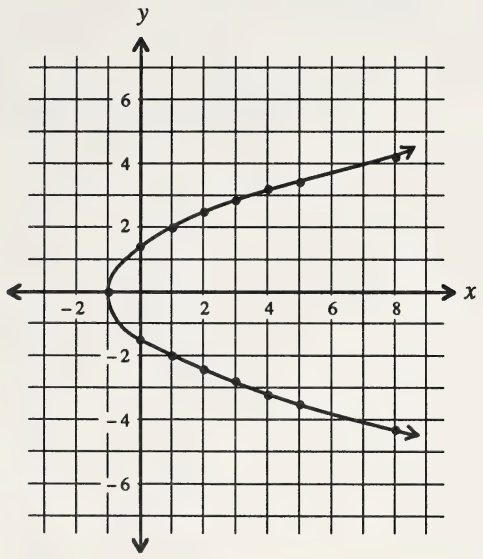
h. Domain: $x \ge 0,\ x \in R$
Range: $y \ge 1,\ y \in R$
Since you cannot have the square root of a negative number, x must be positive or zero. Then the value of y must always be equal to or greater than one.

6. Solve for y.

$$2y^2 - 4x = 4$$
$$2y^2 = 4x + 4$$
$$y^2 = \frac{4x + 4}{2}$$
$$y = \pm\sqrt{2x + 2}$$

Make a table of values using values of x that are greater than or equal to -1. This ensures that no negative value appears under the square root sign.

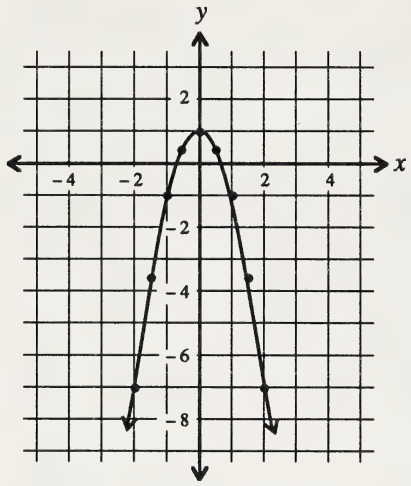
x	y
-1	0
0	$\pm\sqrt{2}$
1	± 2
2	$\pm\sqrt{6}$
3	$\pm\sqrt{8}$



This is not a function. The domain is $x \geq -1$, $x \in R$ and the range is $y \in R$.

7.

x	y
-2	-7
-1	-1
0	1
1	-1
2	-7



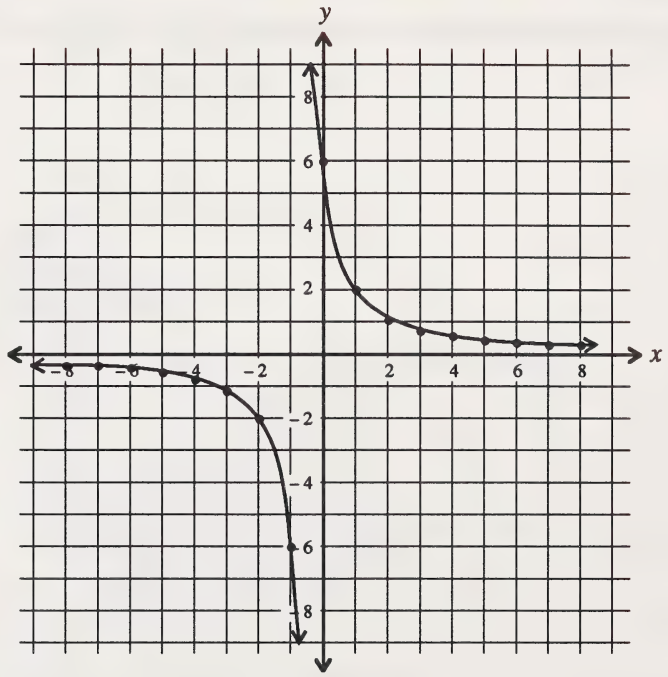
This is a function.
Domain: $x \in R$
Range: $y \leq 1$

8.

x	y
-6	$-\frac{6}{11}$
-4	$-\frac{6}{7}$
-2	-2
-1	-6
$-\frac{3}{4}$	-12
$-\frac{1}{4}$	12
0	6
1	2
2	$\frac{6}{5}$

If $x=-\frac{1}{2}$, then $y=\frac{6}{2\left(-\frac{1}{2}\right)+1}$

$=\frac{6}{0}$ (This is undefined.)



This is a function.

Domain: $x \in R,\ x \neq -\frac{1}{2}$

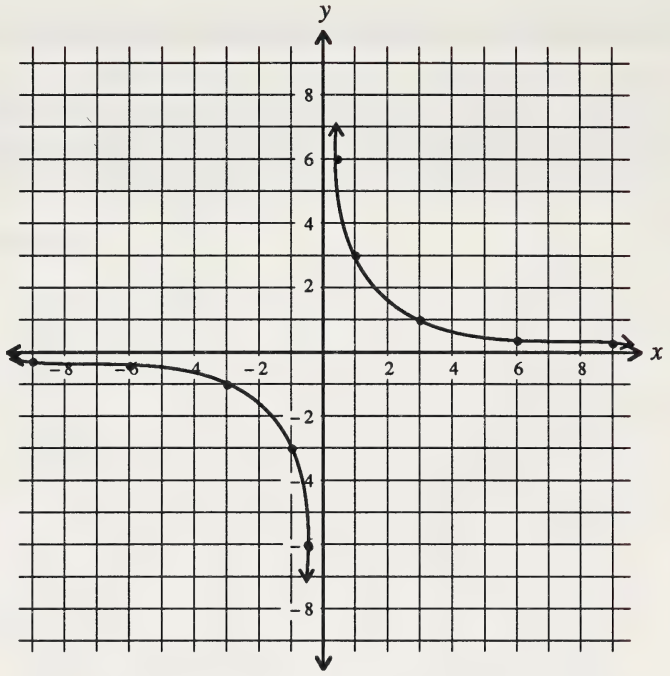
Range: $y \in R,\ y \neq 0$

9. First solve for y.

$$y = \frac{9}{3x}$$

$$y = \frac{3}{x}$$

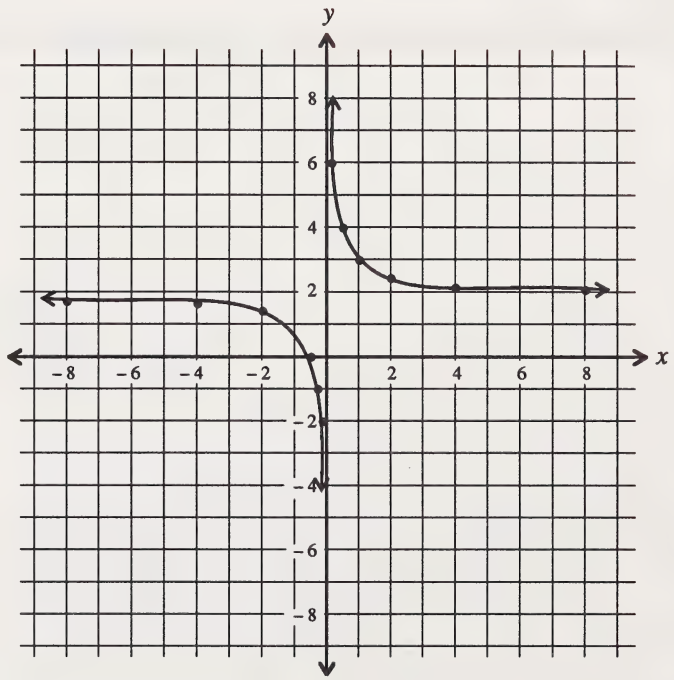
x	y
-9	$-\frac{1}{3}$
-6	$-\frac{1}{2}$
-3	-1
-1	-3
$-\frac{1}{2}$	-6
$-\frac{1}{3}$	-9
$\frac{1}{3}$	9
$\frac{1}{2}$	6
1	3
3	1
6	$\frac{1}{2}$
9	$\frac{1}{3}$



Domain: $x \in R,\ x \neq 0$
Range: $y \in R,\ y \neq 0$
Yes, this is a function.

10.

x	y
-8	$1\frac{7}{8}$
-4	$1\frac{3}{4}$
-2	$1\frac{1}{2}$
-1	1
$-\frac{1}{2}$	0
$-\frac{1}{3}$	-1
$-\frac{1}{4}$	-2
$\frac{1}{4}$	6
$\frac{1}{2}$	4
1	3
2	$2\frac{1}{2}$
4	$2\frac{1}{4}$
8	$2\frac{1}{8}$



Domain: $x \in R,\ x \neq 0$
Range: $y \in R,\ y \neq 2$
This is a function.

Activity 2

Write functions using functional notation and evaluate them by substitution for the domain.

1. a. $f(-8)=\dfrac{2(-8)}{-8-6}$

$f(-8)=\dfrac{-16}{-14}$

$f(-8)=\dfrac{8}{7}$

$f(-8)=1\dfrac{1}{7}$

b. $f(0)=\dfrac{2(0)}{0-6}$

$f(0)=\dfrac{0}{-6}$

$f(0)=0$

c. $f\left(x^2\right)=\dfrac{2\left(x^2\right)}{x^2-6}$

$f\left(x^2\right)=\dfrac{2x^2}{x^2-6}$

Notice that this cannot be simplified further.

d. $f(-1)+f(3)=\dfrac{2(-1)}{-1-6}+\dfrac{2(3)}{3-6}$

$=\dfrac{-2}{-7}+\dfrac{6}{-3}$

$=\dfrac{2}{7}-2$

$=-1\dfrac{5}{7}$

2. a. $g(-1)=2(-1)^3-(-1)^2+5(-1)-1$

$=2(-1)-(1)+5(-1)-1$

$=-2-1-5-1$

$=-9$

b. $g(2)=2(2)^3-(2)^2+5(2)-1$

$=2(8)-(4)+5(2)-1$

$=16-4+10-1$

$=21$

c. $g(0)=2(0)^3-(0)^2+5(0)-1$

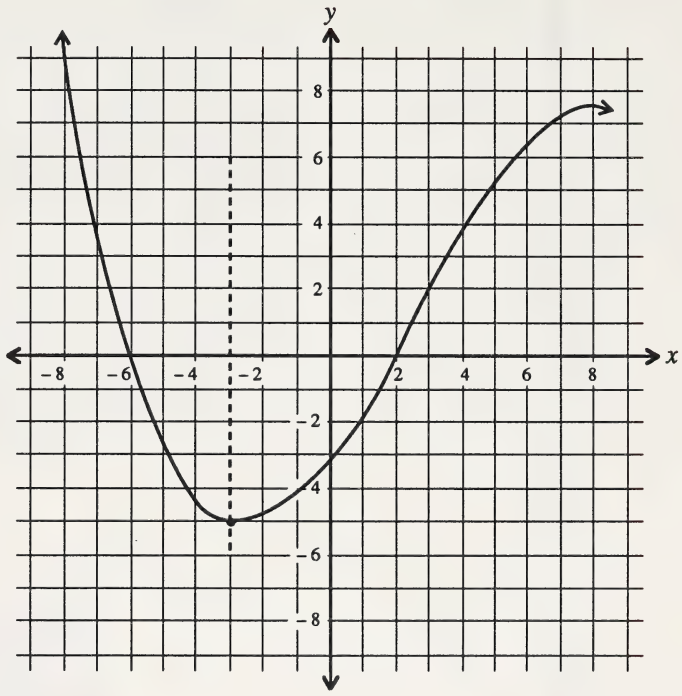
$=0-0+0-1$

$=-1$

d. $g(a)=2(a)^3-a^2+5(a)-1$

$=2a^3-a^2+5a-1$

3.



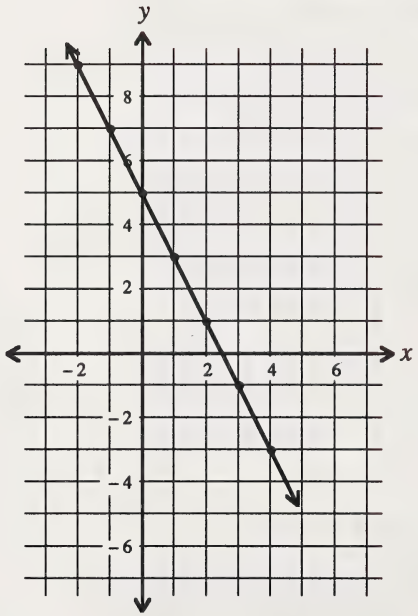
From the graph, $h(-3)=-5$.

Activity 3

Graphically and algebraically illustrate and define different kinds of functions, in particular, polynomial (linear, quadratic, and cubic), reciprocal, absolute-value, square-root, and exponential functions.

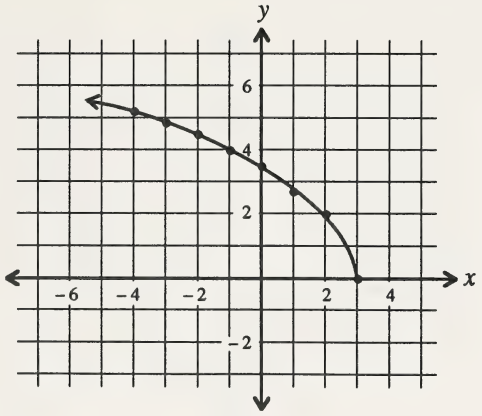
1. a. linear function $g(x) = -2x + 5$

x	y
-2	9
-1	7
0	5
1	3
2	1
3	-1
4	-3



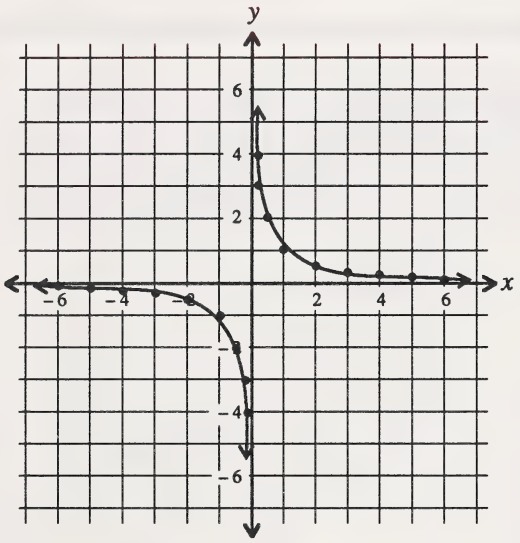
b. square-root function $Q(x) = 2\sqrt{3-x}$

x	y	
-4	$2\sqrt{7}$	$(\doteq 5.3)$
-3	$2\sqrt{6}$	$(\doteq 4.9)$
-2	$2\sqrt{5}$	$(\doteq 4.5)$
-1	4	
0	$2\sqrt{3}$	$(\doteq 3.5)$
1	$2\sqrt{2}$	$(\doteq 2.8)$
2	2	
3	0	



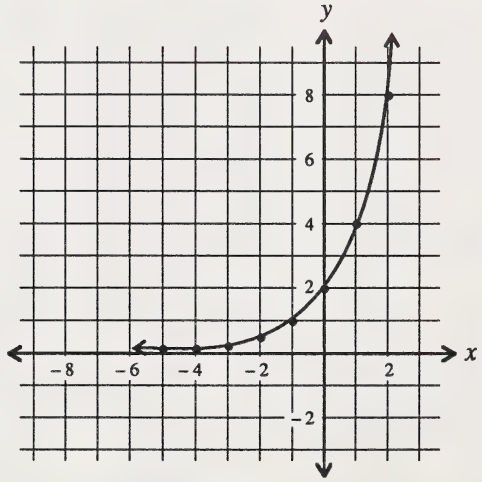
c. reciprocal function $P(x) = \frac{1}{x}$

x	y	x	y
-6	$-\frac{1}{6}$	6	$\frac{1}{6}$
-5	$-\frac{1}{5}$	5	$\frac{1}{5}$
-4	$-\frac{1}{4}$	4	$\frac{1}{4}$
-3	$-\frac{1}{3}$	3	$\frac{1}{3}$
-2	$-\frac{1}{2}$	2	$\frac{1}{2}$
-1	-1	1	1
$-\frac{1}{2}$	-2	$\frac{1}{2}$	2
$-\frac{1}{3}$	-3	$\frac{1}{3}$	3
$-\frac{1}{4}$	-4	$\frac{1}{4}$	4



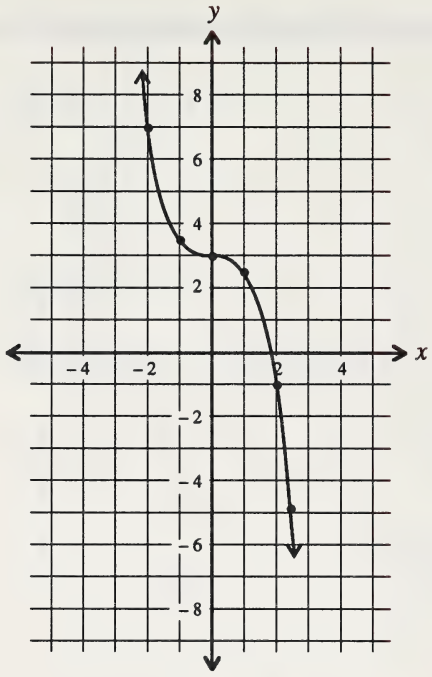
x	y
−5	$\frac{1}{16}$
−4	$\frac{1}{8}$
−3	$\frac{1}{4}$
−2	$\frac{1}{2}$
−1	1
0	2
1	4
2	8
3	16

d. exponential function $f(x) = 2^{x+1}$



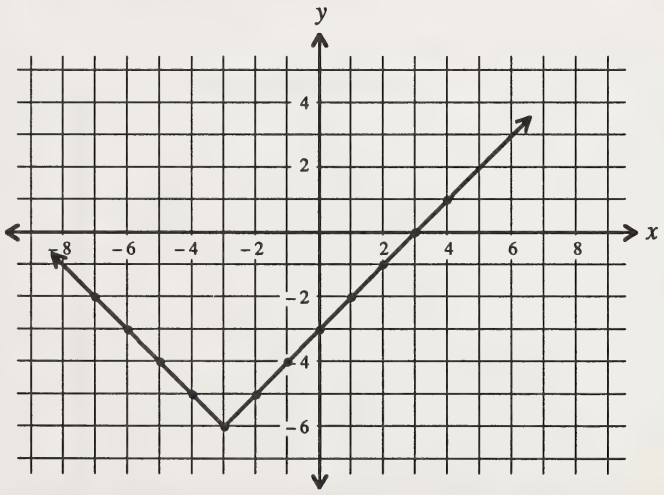
e. cubic function $f(x) = -\frac{1}{2}x^3 + 3$

x	y
−2	7
−1	3.5
0	3
1	2.5
2	−1
2.5	−4.8



f. absolute-value function $y=|x+3|-6$

x	y
-7	-2
-6	-3
-5	-4
-4	-5
-3	-6
-2	-5
-1	-4
0	-3
1	-2
2	-1
3	0
4	1



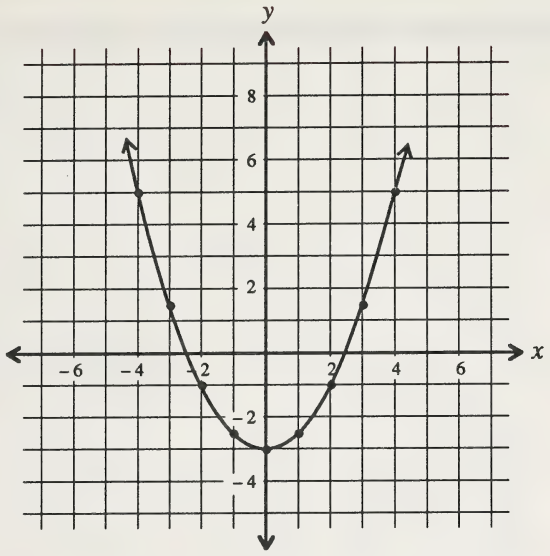
g. quadratic function

i. $y=\frac{1}{2}x^2-3$

x	y
-4	5
-3	$1\frac{1}{2}$
-2	-1
-1	$-2\frac{1}{2}$
0	-3
1	$-2\frac{1}{2}$
2	-1
3	$1\frac{1}{2}$
4	5

The following is a sample calculation at $x=-3$.

$$y=\frac{1}{2}(-3)^2-3$$
$$=\frac{1}{2}(9)-3$$
$$=4\frac{1}{2}-3$$
$$=1\frac{1}{2}$$

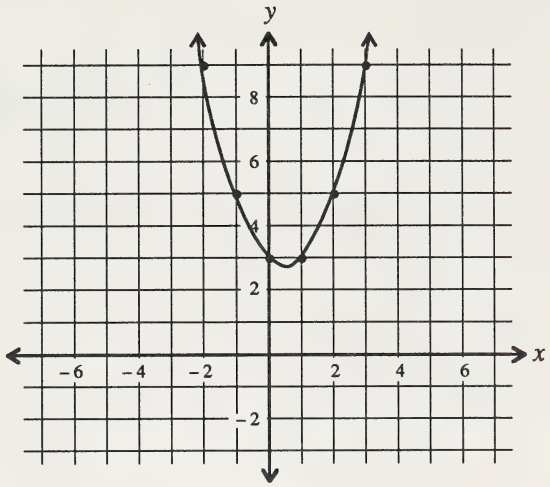


ii. $g(x) = x^2 - x + 3$

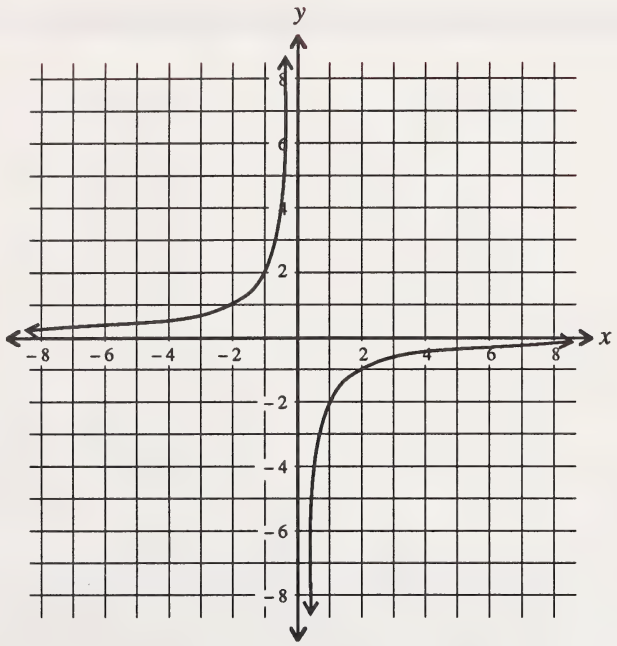
x	y
-2	9
-1	5
0	3
1	3
2	5
3	9

The following is a sample calculation at $x = -1$.

$$\begin{aligned} g(-1) &= (-1)^2 - (-1) + 3 \\ &= 1 + 1 + 3 \\ &= 5 \end{aligned}$$

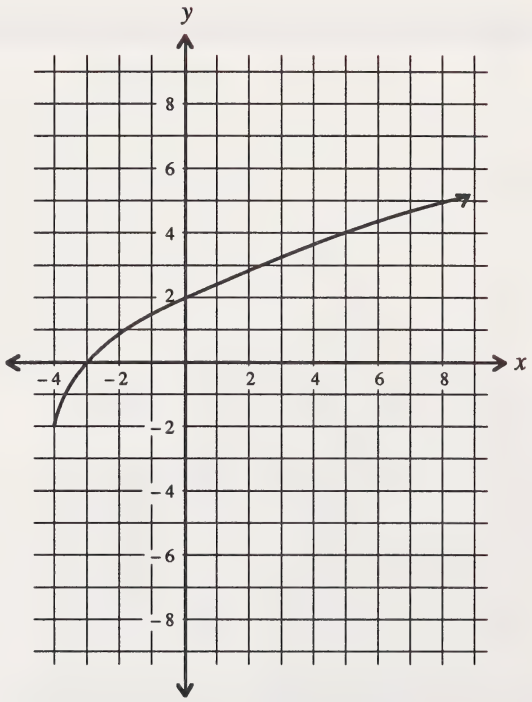


2. a.



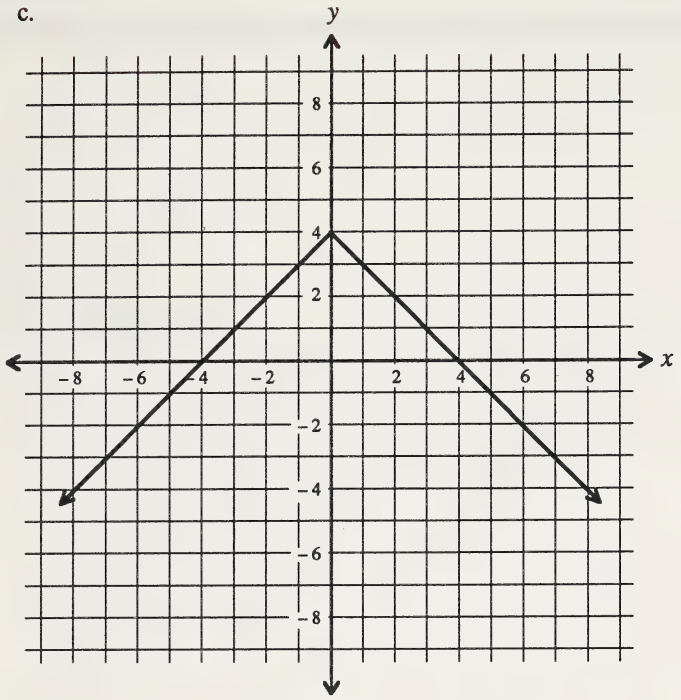
reciprocal function

b.



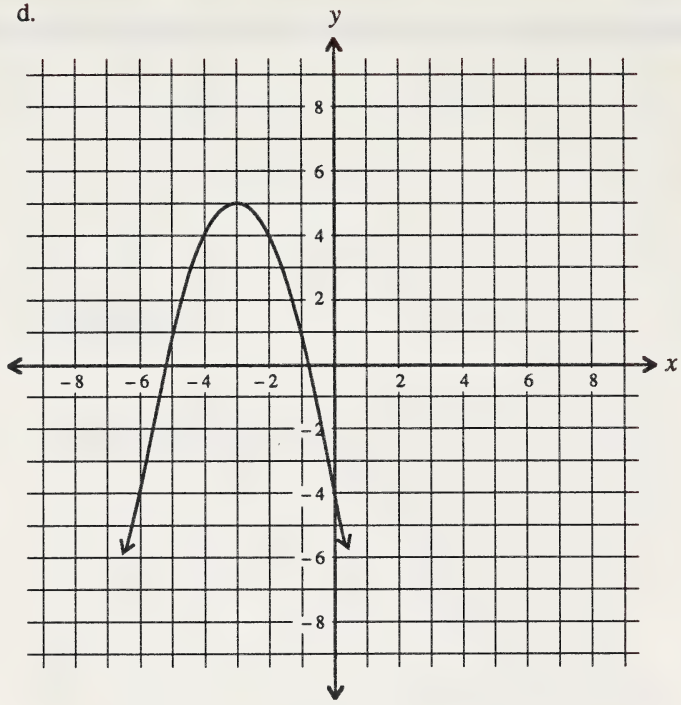
square-root function

c.



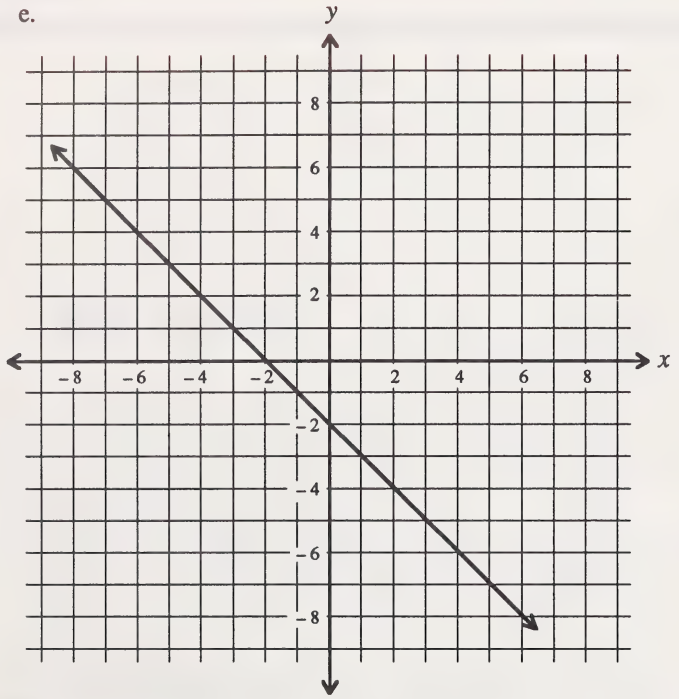
absolute-value function

d.



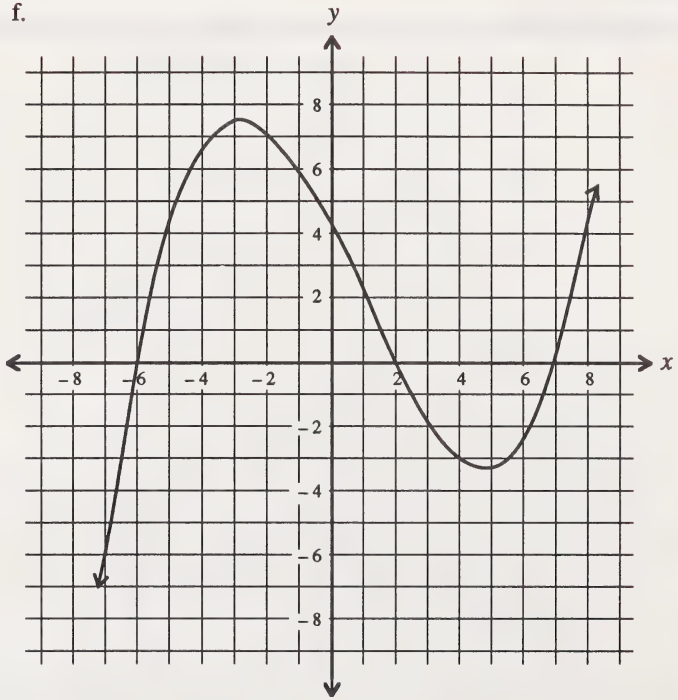
quadratic function

e.



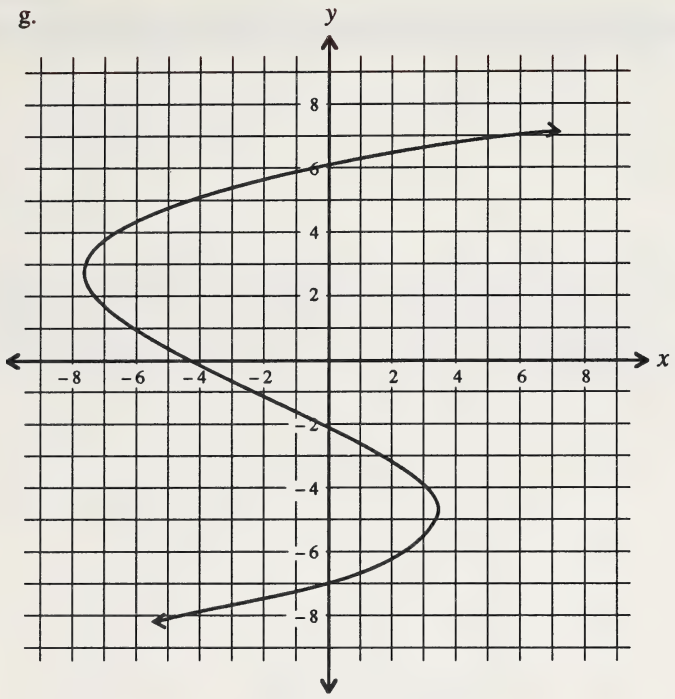
linear function

f.



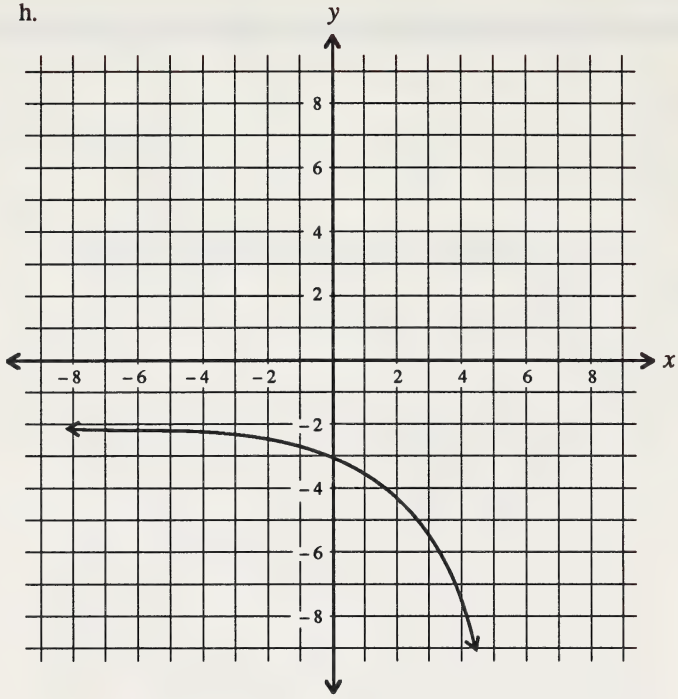
cubic function

g.



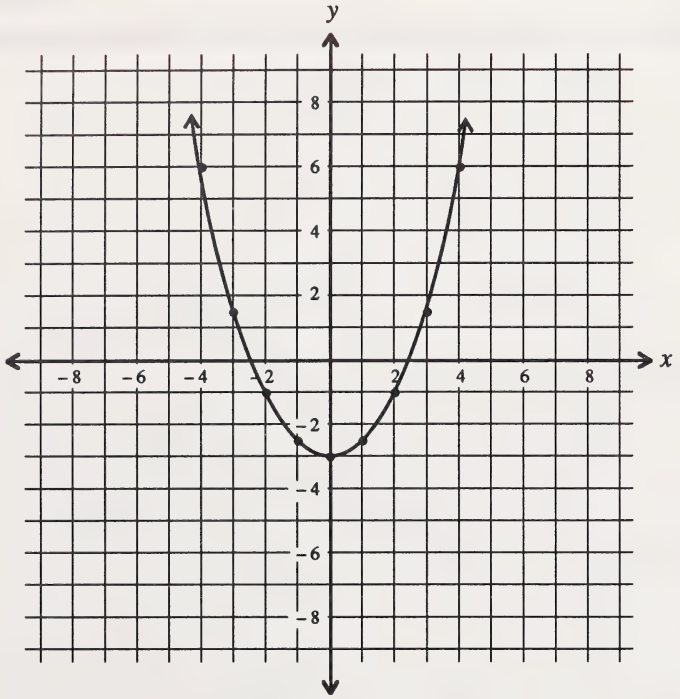
not a function

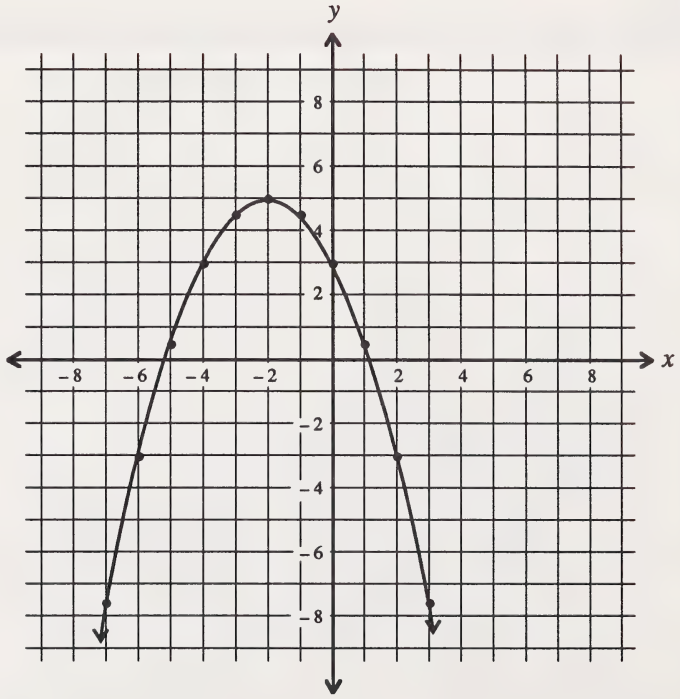
h.



exponential function

3.





One opens upwards, and the other opens downwards.

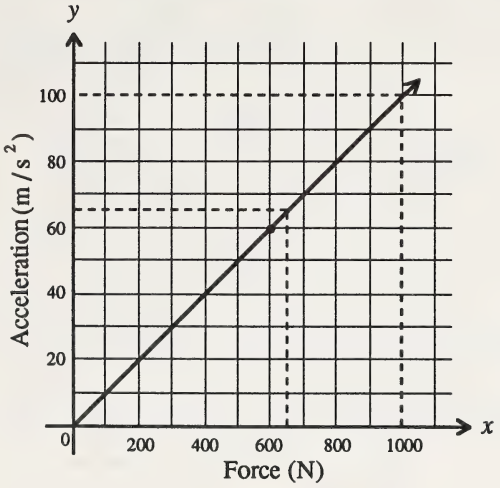
4. The general equation for a reciprocal function is $y = \frac{1}{x}$.

5. For an exponential function the graph has no starting or ending point. The graph extends in one direction approaching a certain value of y and increases indefinitely in the other direction. For a square-root function the graph starts at a fixed point and continues in one direction.

Activity 4

Solve problems involving functions algebraically and by drawing, interpolating, extrapolating, interpreting, and analyzing graphs of functions.

1. a.



b. From the graph $a \doteq 65 \text{ m/s}^2$ when $F = 650$ N and $m = 10$ kg.

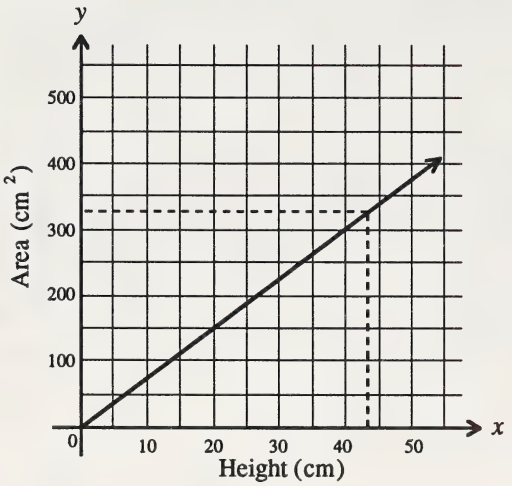
c. From the graph $F \doteq 1000$ N when $a = 100 \text{ m/s}^2$ and $m = 10$ kg.

2. a. $A = \frac{1}{2}bh$

b. $A = \frac{1}{2}(15 \text{ cm})(23 \text{ cm})$

$= 172.5 \text{ cm}^2$

c.

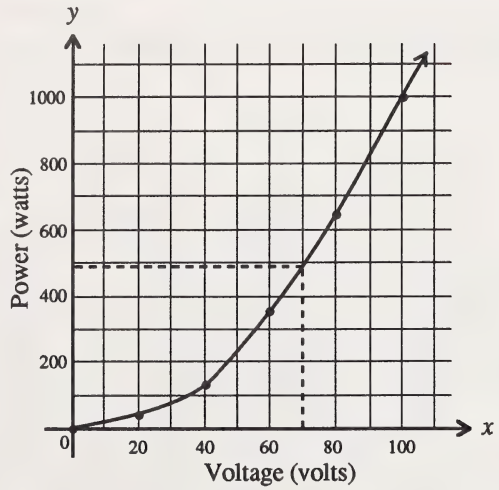


d. From the graph $A \doteq 322 \text{ cm}^2$ when the base is 15 cm and the height is 43 cm.

e. This graph represents a linear function.

3. a. This is a quadratic function since the degree is two.

b.



c. When $V = 70$ V, $P \doteq 490$ W.

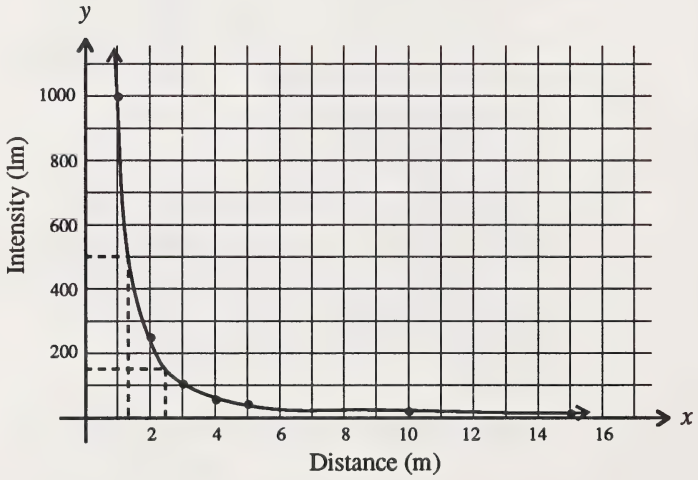
4. a. $I = \dfrac{1000}{15^2}$

$= \dfrac{1000}{225}$

$= 4.44$ lm (lumens)

b.

d (m)	I (lm)
1	1000.0
2	250.0
3	111.1
4	62.5
5	40.0
10	10.0
15	4.4

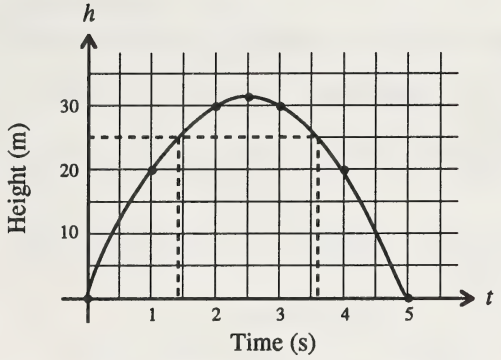


c. From the graph $I = 160$ lm at 2.50 m from the source.

d. From the graph $d = 1.4$ m from the source at the point where light intensity is 500 lm.

5. a.

t	h
0.0	0.00
1.0	20.00
2.0	30.00
2.5	31.25
3.0	30.00
4.0	20.00
5.0	0.00

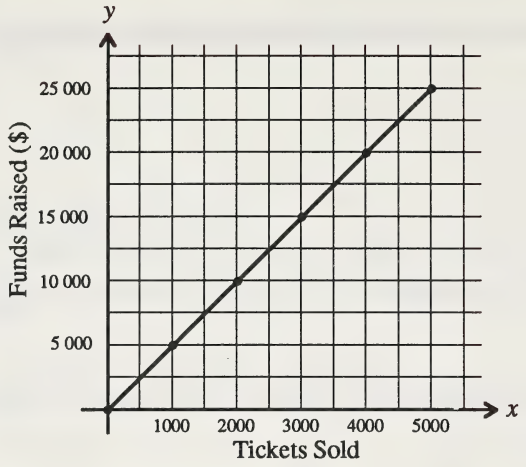


b. From the graph the maximum height is 31.25 m.

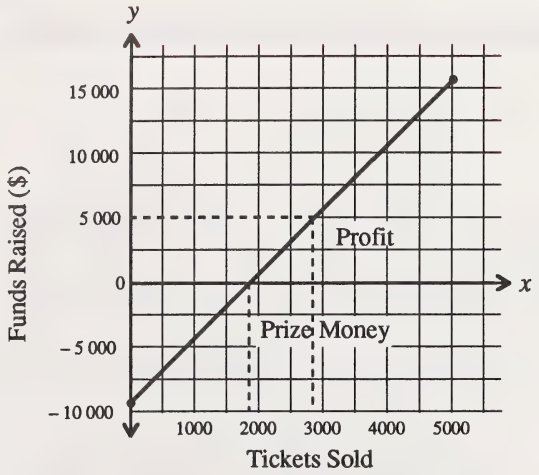
c. From the graph the total time the ball is in the air is 5 s.

d. From the graph the two times the ball is at the height of 25 m are 1.4 s and 3.6 s.

6. a.



b. Here is another way to draw the graph. Since $9000 must be paid in prizes, let the $9000 level be zero which is the point at which the fund-raiser breaks even. All funds raised above $9000 are actual profit.



According to the graph, 2800 tickets must be sold to raise $5000.

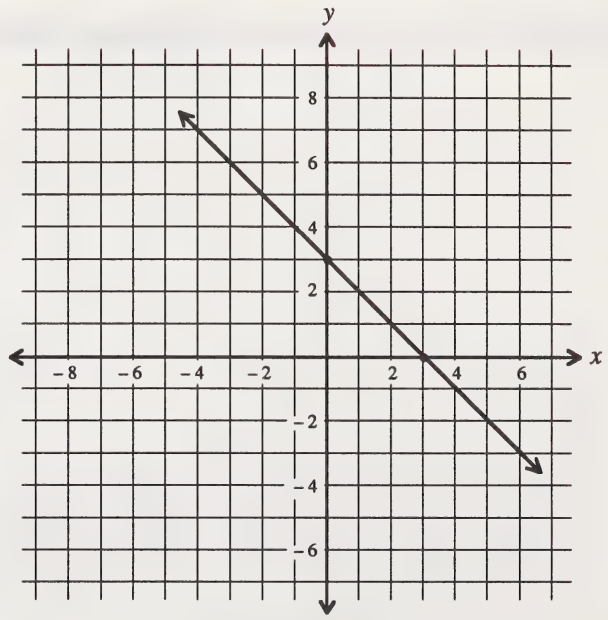
c. The largest amount of money that can be raised is $16 000.

d. To break even, 1800 tickets must be sold.

e. The domain is from 0 to 5000 with only whole numbers, and the range is $0 to $25 000.

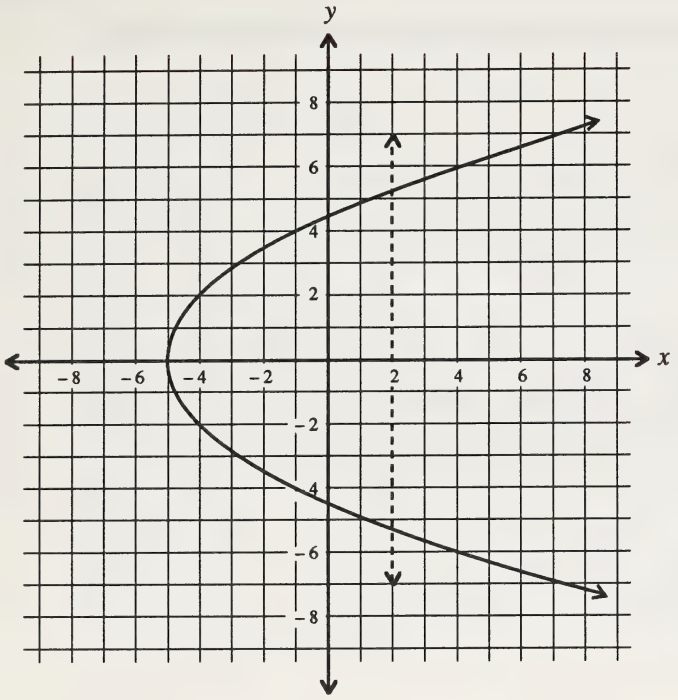
Extra Help

1. a. The equation $y = 3 - x$ is a function. You cannot find two points that have the same first coordinate. When the graph of this function is drawn, you cannot draw a vertical line so that it crosses this graph twice.



b. This set of points is not a function since the points $(2, 5)$ and $(2, -5)$ have the same first coordinate.

c. This is not a function since a vertical line through the graph will pass through two points. See the following graph for one example.



d. This is a graph of a function. While this graph may seem to have a section where the graph looks like a vertical line, each point is unique in that no two points have the same x-value.

2. a. $f(-3)=(-3)^2-20$
$=9-20$
$=-11$

b. $g(5)=(5)^3-(5)^2+3(5)$
$=125-25+15$
$=115$

c. $P(0)=3(0)^4-5(0)^3+2$
$=2$

Extensions

1. $f(g(x))=(5x+1)^2-7(5x+1)$
$=25x^2+10x+1-35x-7$
$=25x^2-25x-6$

$g(f(x))=5(x^2-7x)+1$
$=5x^2-35x+1$

2. $f(g(x))=3(x^2-1)$
$=3x^2-3$

$g(f(x))=(3x)^2-1$
$=9x^2-1$

3. $f(g(x))=\dfrac{1}{2(5x+2)-1}$
$=\dfrac{1}{10x+4-1}$
$=\dfrac{1}{10x+3}$

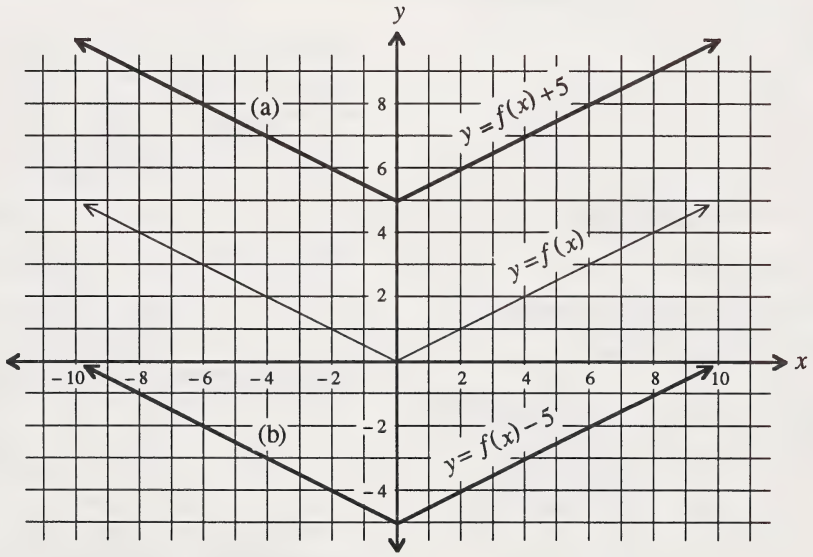
$g(f(x))=5\left(\dfrac{1}{2x-1}\right)+2$
$=\dfrac{5}{2x-1}+2$

Exploring Topic 3

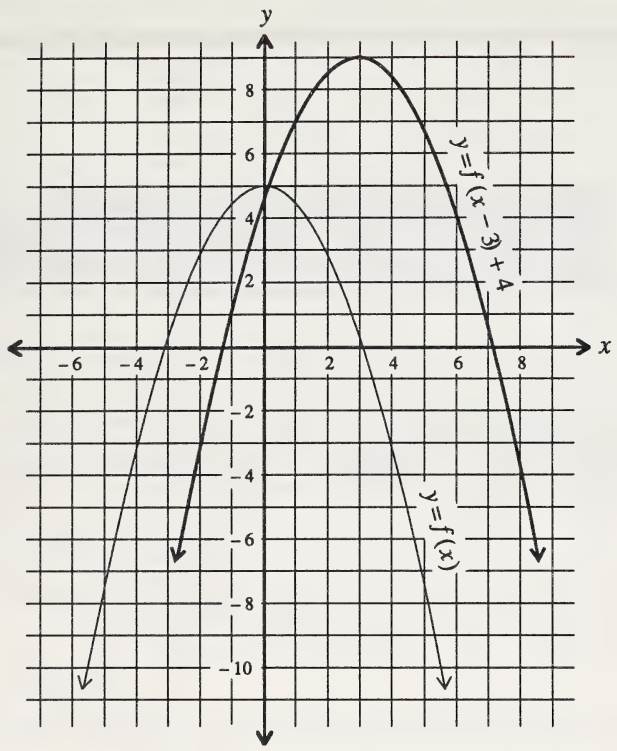
Activity 1

Investigate and describe the transformation effects of the parameters a and b in $y = f(x-a)+b$ on the graph of $y = f(x)$.

1.



2. a.



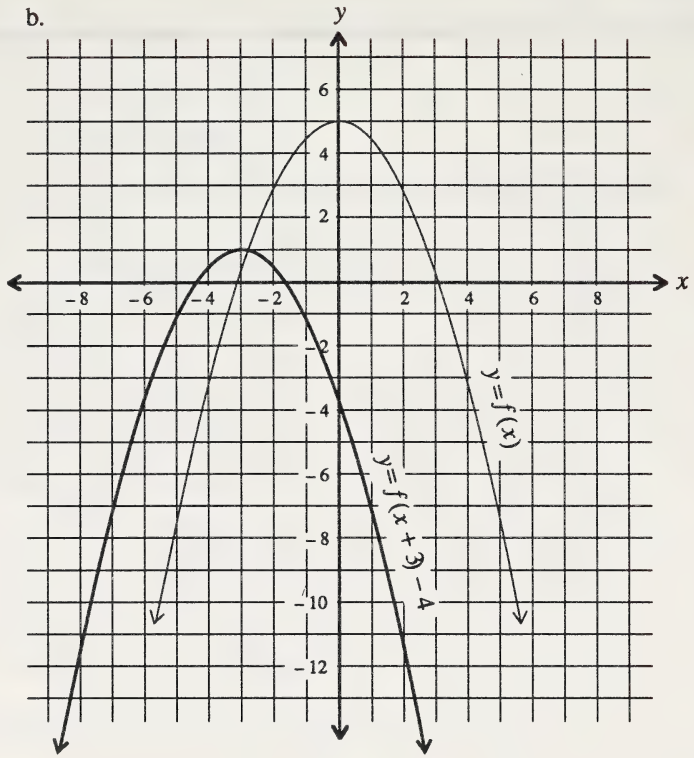
$y = f(x-3)+4$

$a = 3$

$b = 4$

This graph will be the graph of $y = f(x)$ moved three units to the right and four units up.

b.



$y = f(x+3)-4$

$a = -3$

$b = -4$

This graph will be the graph of $y = f(x)$ moved three units to the left and four units downward.

3. a. $y = 2^x + 3$ — In order for the graph to move up, b has to equal 3.

b. $y = 2^x - 5$ — In order for the graph to move down, b has to equal -5.

4. a. $y = |x+3+5| + 4$
$= |x+8| + 4$
$a = -3$ and $b = 4$

b. $y = |x-4+5| - 3$
$= |x+1| - 3$
$a = 4$ and $b = -3$

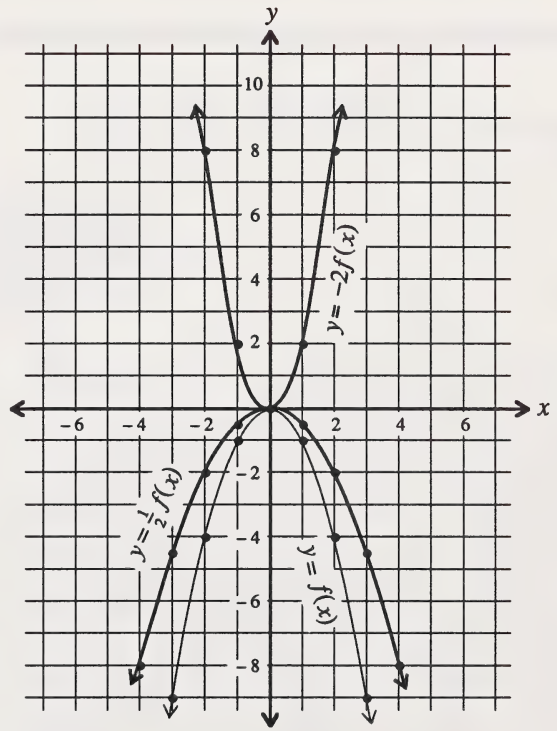
5. a. $y = (x-6)^3 - 5 - 5$
$= (x-6)^3 - 10$
$a = 6$ and $b = -5$

b. $y = (x+7)^3 - 5 + 2$
$= (x+7)^3 - 3$
$a = -7$ and $b = 2$

Activity 2

Investigate and describe the transformation effects of the parameter c in $y = cf(x)$ on the graph of $y = f(x)$.

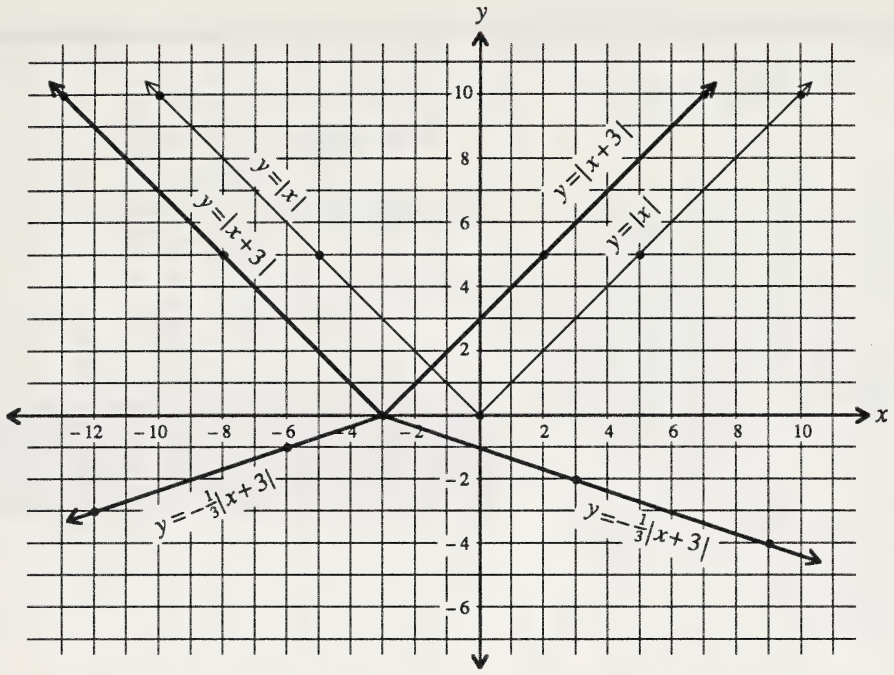
1.



a. The graph of $-2f(x)$ is reflected in the x-axis and is stretched vertically by a factor of 2.

b. The graph of $\frac{1}{2}f(x)$ is compressed vertically by a factor of $\frac{1}{2}$ of the original.

2.



a. Since x is replaced by $x+3$, the graph is moved three units to the left as predicted by $f(x-a)$.

b. The graph is not only moved three units to the left, but it is reflected in the x-axis and compressed vertically by a factor of $\frac{1}{3}$ of the original due to the value of c which is $-\frac{1}{3}$.

3. a. $y=\left(\frac{1}{3}\right)\left(2^{x}\right)$

In order to compress the graph vertically by a factor of $\frac{1}{3}$ of the original, you multiply the function by $\frac{1}{3}$.

b. $y=(3)\left(2^{x}\right)$

4. a. $y=-2^{3x}$

b. $y=\left(-\frac{1}{2}\right)\left(2^{3x}\right)$

When $c<0$, the graph is reflected in the x-axis.

Activity 3

Describe and draw the graphs of $y=cf(x-a)+b$.

1. a. $y=\frac{1}{2}\sqrt{x-5}-3$

When $c=\frac{1}{2}$, the graph is compressed vertically by a factor of $\frac{1}{2}$ of the original.
When $a=5$, the graph moves to the right five units.
When $b=-3$, the graph moves down three units.

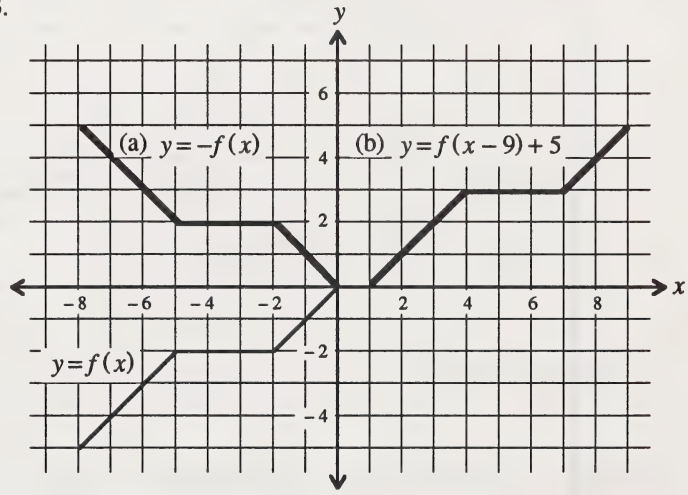
b. $y=2\sqrt{x+5}+3$

When $c=2$, the graph stretches vertically by a factor of 2.
When $a=-5$, the graph moves five units to the left.
When $b=3$, the graph moves up three units.

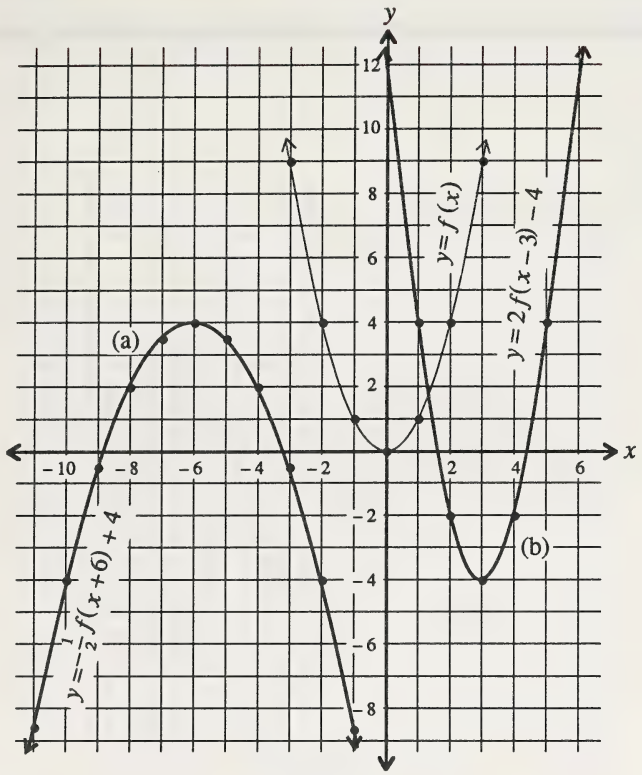
2. a. $y=-\frac{1}{3}|x+3|-7$

b. $y=3|x-3|+7$

3.

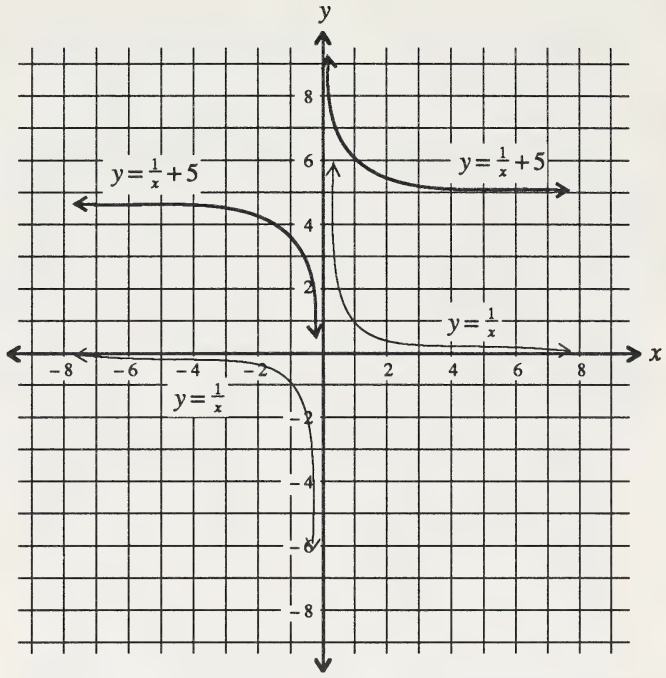


4.

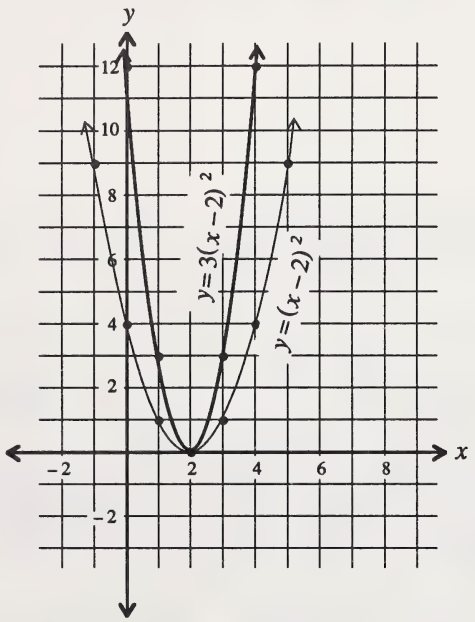


Extra Help

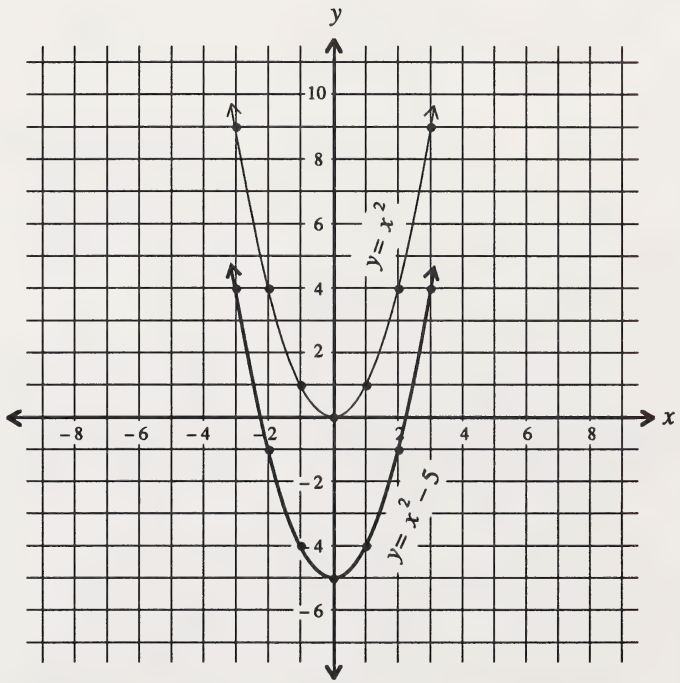
1. The function $y = \frac{1}{x} + 5$ is five units vertically up from $y = \frac{1}{x}$.



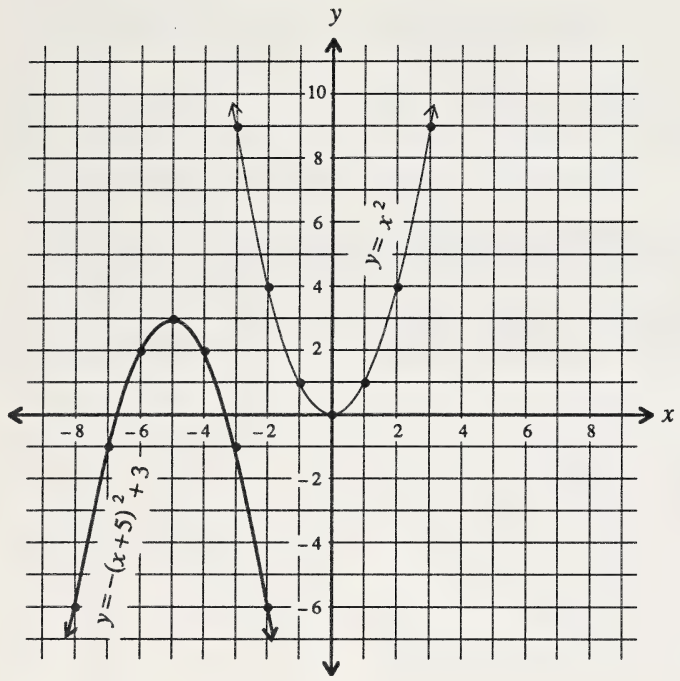
2. The graph of $y = 3(x-2)^2$ is narrower than the graph of $y = (x-2)^2$.



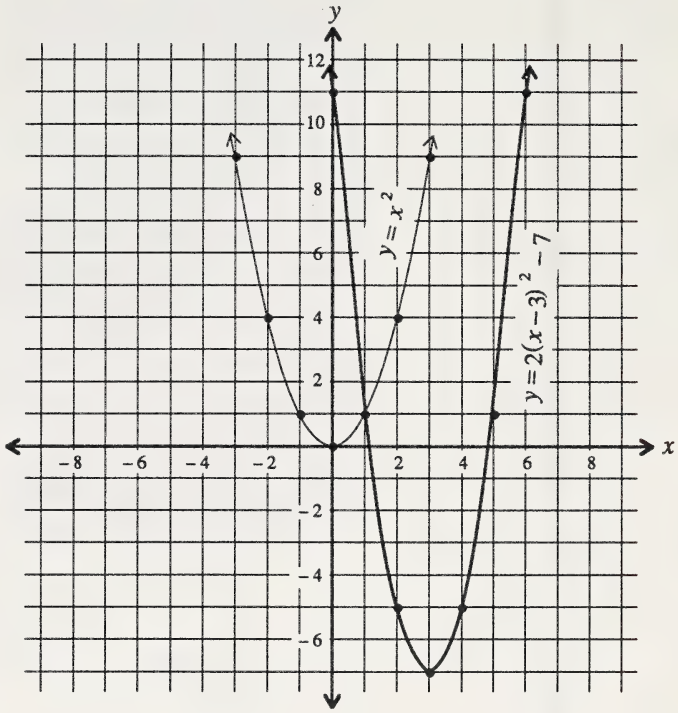
3. The graph of $y = x^2 - 5$ is five units lower than the graph of $y = x^2$.



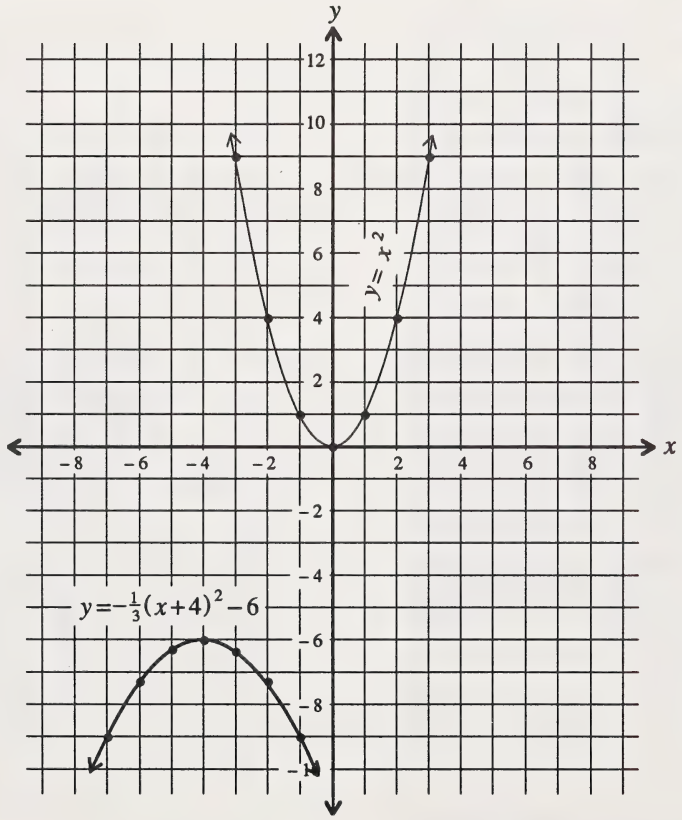
4. The graph of $y=-(x+5)^2+3$ is reflected in the x-axis, and it is five units to the left and three units higher than the graph of $y=x^2$.



5. The graph of $y=2(x-3)^2-7$ is three units to the right of $y=x^2$, seven units lower than $y=x^2$, and stretched vertically by a factor of 2.



6. The graph of $y = -\frac{1}{3}(x+4)^2 - 6$ is reflected in the graph of $y = x^2$. The new graph is moved to the left four units and downwards six units of $y = x^2$. It has also been compressed vertically by a factor of $\frac{1}{3}$ of the original.



Extensions

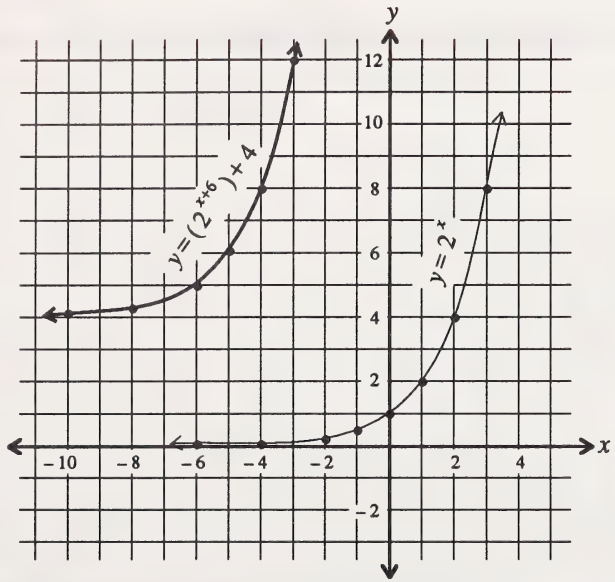
1. a. No, $y=\left(2^{x+6}\right)+4$ is the same shape as $y=2^x$.

   b. Yes, $y=\left(2^{x+6}\right)+4$ is six units to the left of $y=2^x$.

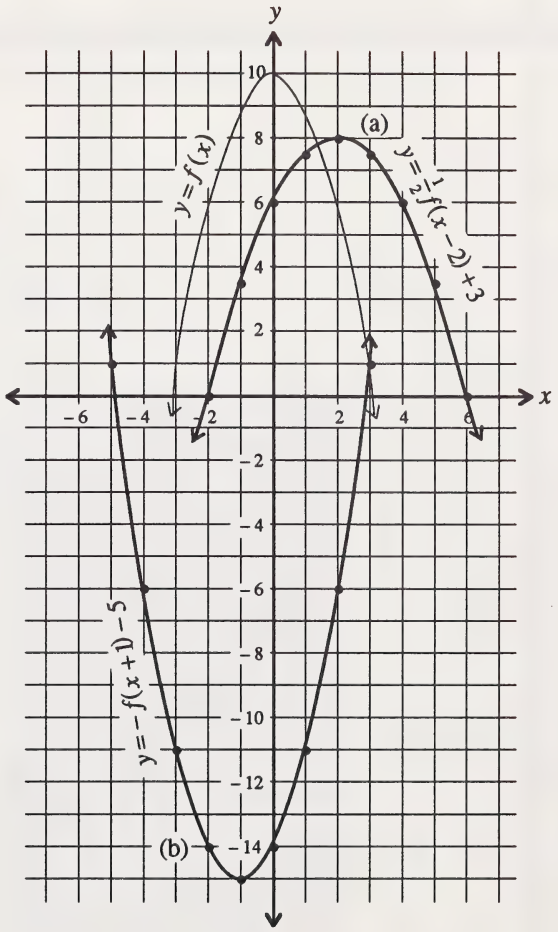
   c. Yes, the graph of $\left(2^{x+6}\right)+4$ gets very close to 4 as x goes to the left and $y=2^x$ gets very close to 0 as x goes to the left.
   Also, the point $(0,1)$ corresponds to the point $(-6,5)$ which is four units higher.

Note: The blanks in the table represent large numbers.

x	$y=2^x$	$y=\left(2^{x+6}\right)+4$
-8	$\frac{1}{256}$	$4\frac{1}{4}$
-6	$\frac{1}{64}$	5
-5	$\frac{1}{32}$	6
-4	$\frac{1}{16}$	8
-3	$\frac{1}{8}$	12
-2	$\frac{1}{4}$	20
-1	$\frac{1}{2}$	
0	1	
1	2	
2	4	
3	8	



2.



If you did not understand the graphs, do not worry. The transformations do not move vertically b units since the vertex of the function $y=f(x)$ is not on the x-axis. The function $f(x)=10-x^2$ was the function graphed. Use this function to make a table of values for the transformations and plot them on the graph. The answers are given as follows.

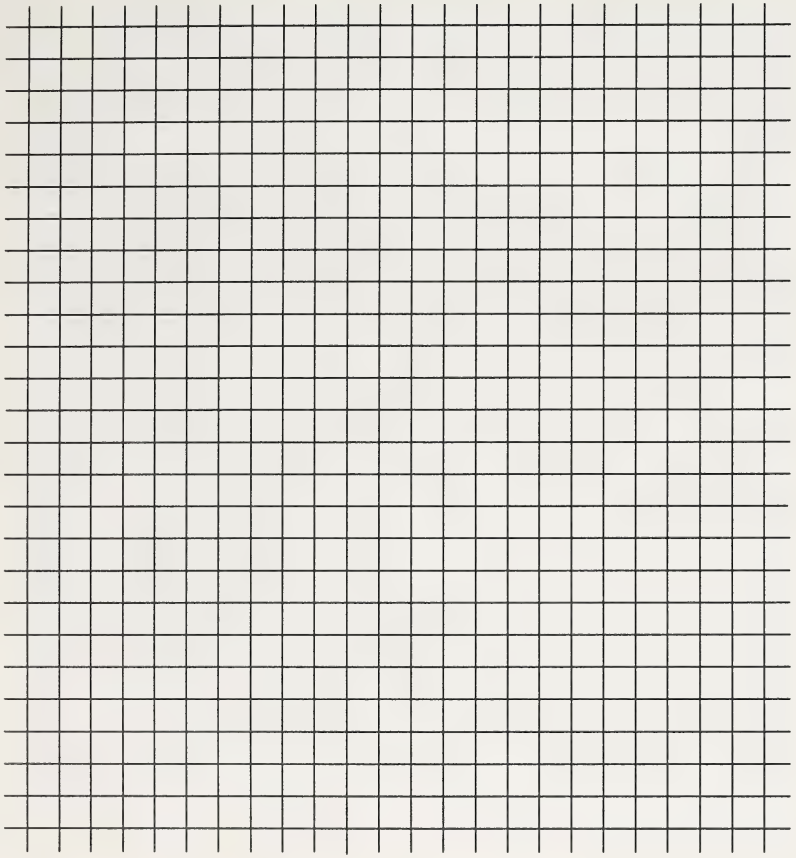
x	$y=\frac{1}{2}\left(10-(x-2)^2\right)+3$	$y=-\left(10-(x+1)^2\right)-5$
-5	-16.5	1
-4	-10.0	-6
-3	-4.5	-11
-2	0.0	-14
-1	3.5	-15
0	6.0	-14
1	7.5	-11
2	8.0	-6
3	7.5	1
4	6.0	10
5	3.5	
6	0.0	
7	-4.5	

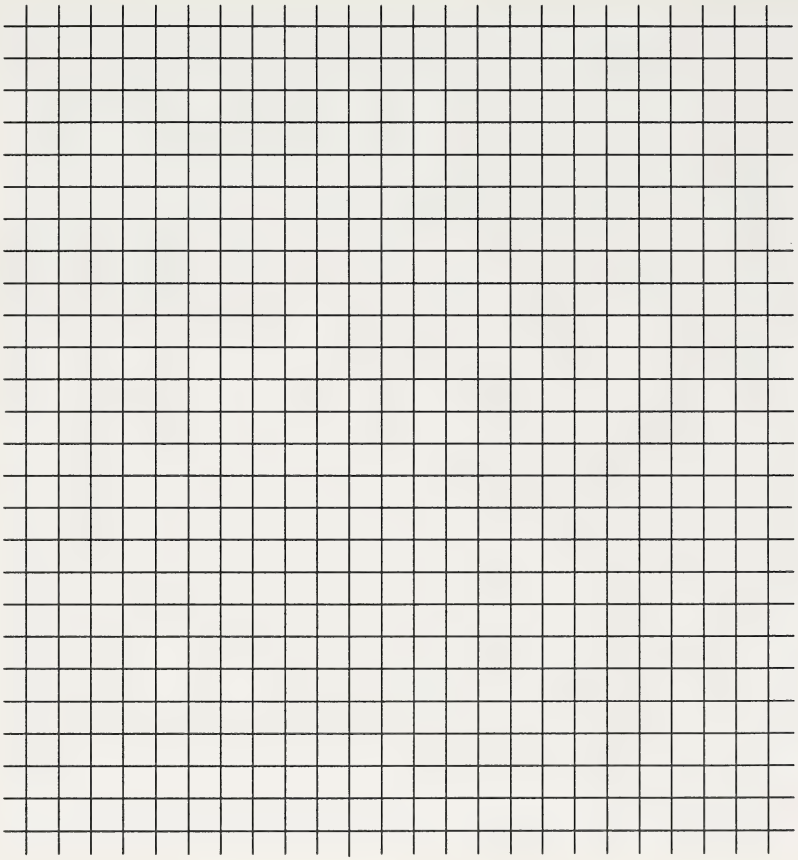
Examine these transformations to see if you can make further predictions. Begin with $y=f(x)$ transformed to $y=\frac{1}{2}f(x-2)+3$. The graph moved two units to the right as predicted. The value $c=\frac{1}{2}$ means the graph is made wider so the original y-value is reduced by $\frac{1}{2}$. The vertex must move down $\frac{1}{2}(10)=5$ units. From this point move up b units (three units) which is eight units above the x-axis.

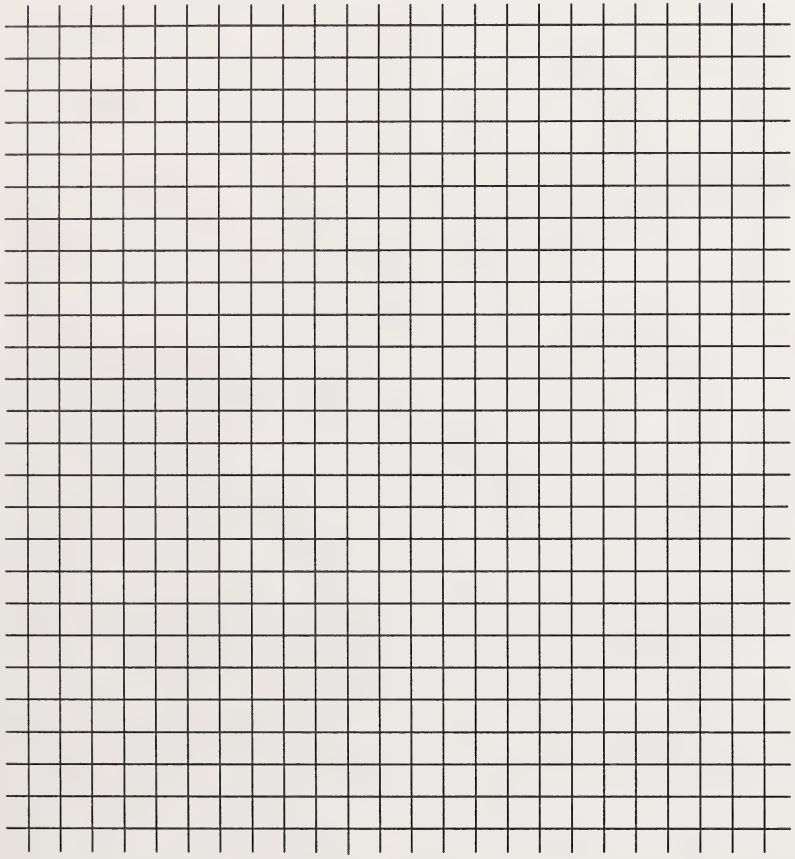
Try this with the next transformation. Since $c=-1$, invert $f(x)$ so that the new vertex is at $(0,-10)$. Move down five units $(b=-5)$ and one unit to the left $(a=-1)$. The vertex should be at $(-1,-15)$, and it is.

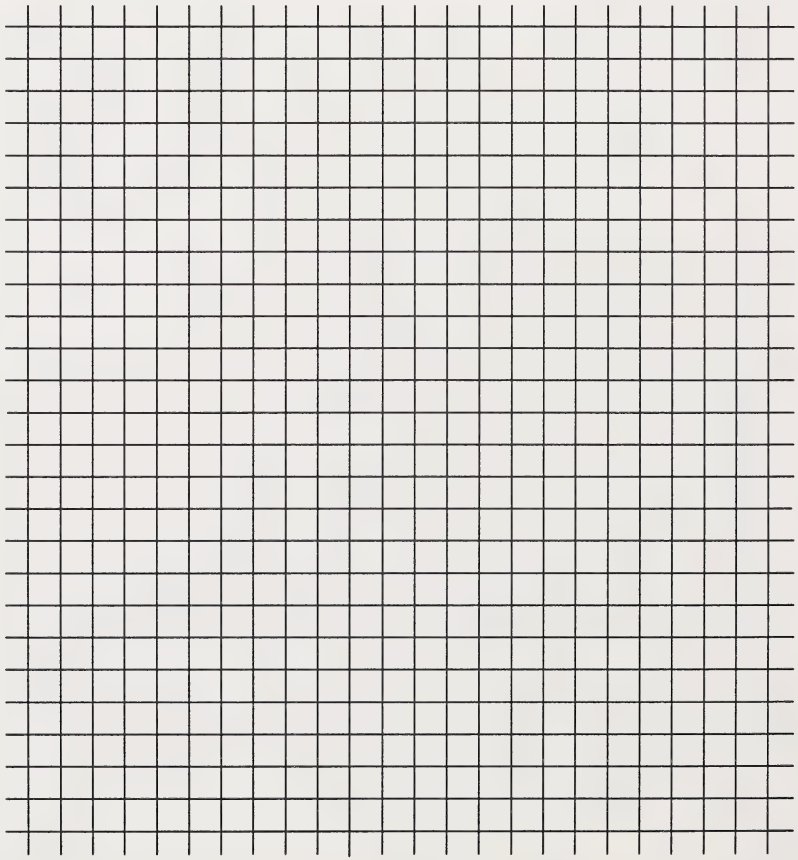
Appendix B
Graphing Material

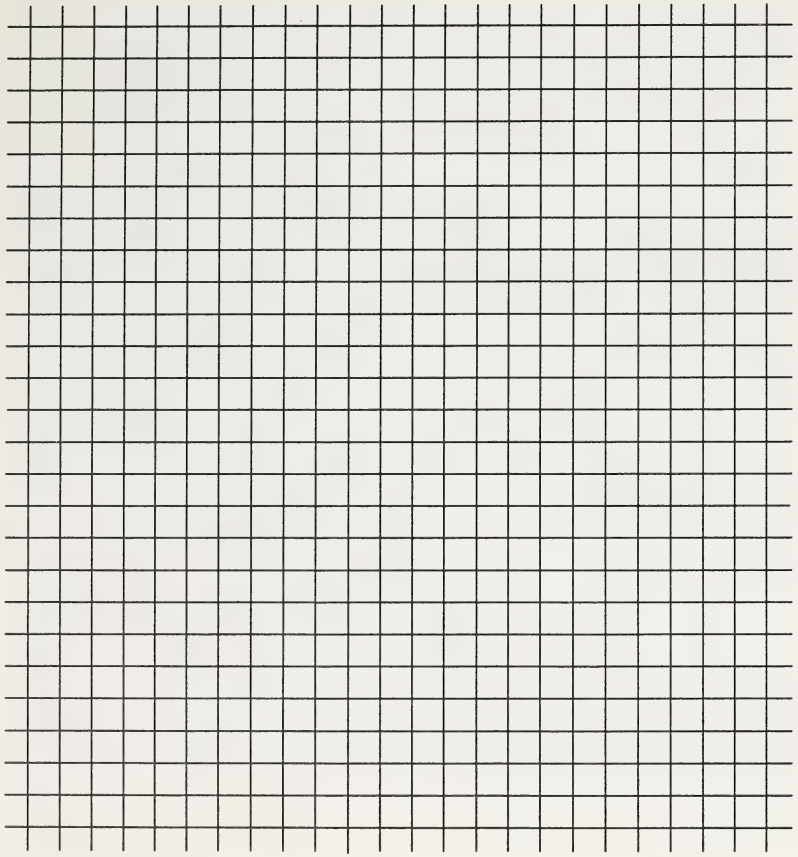
Graph Paper

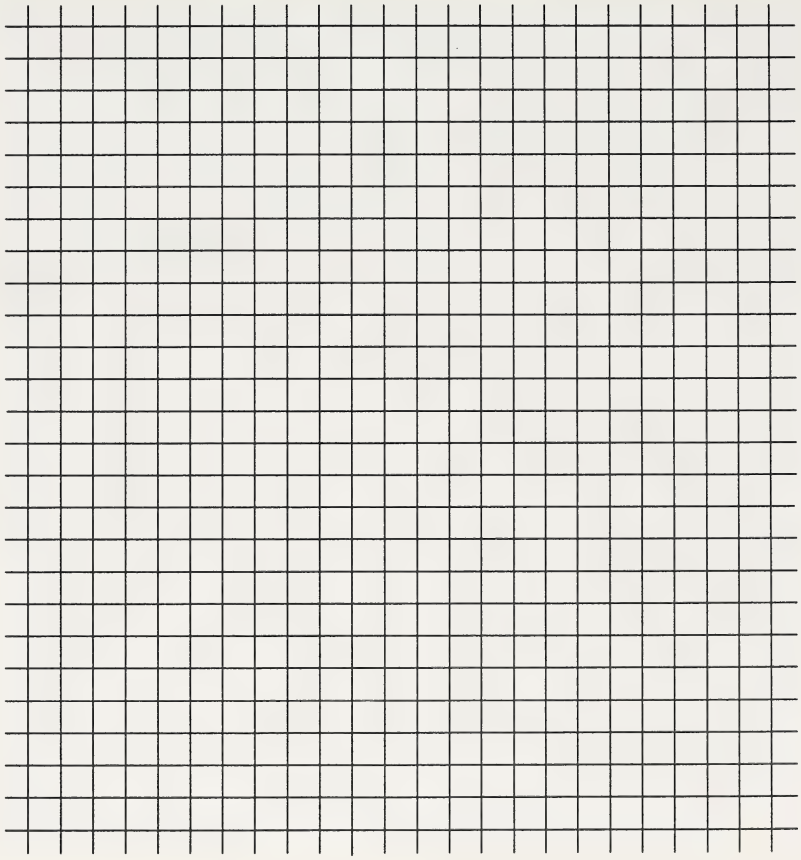


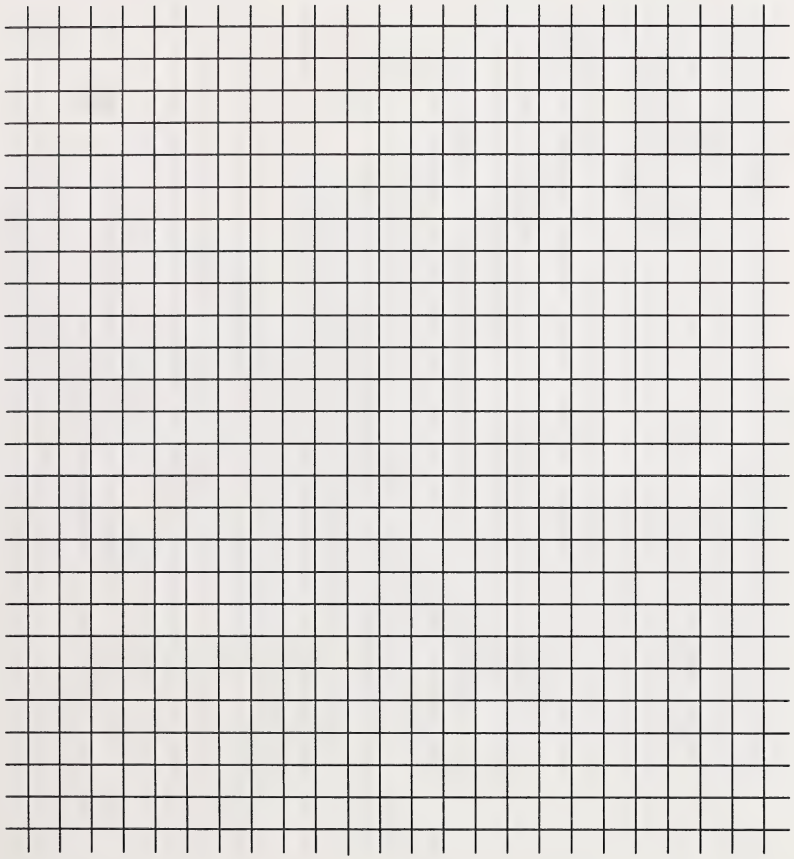


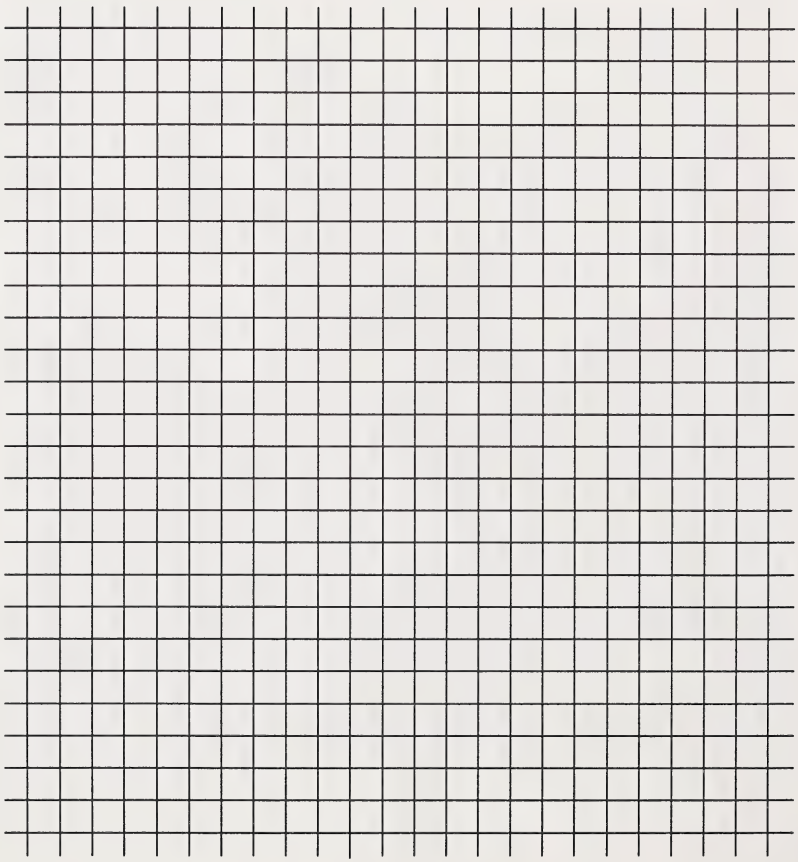


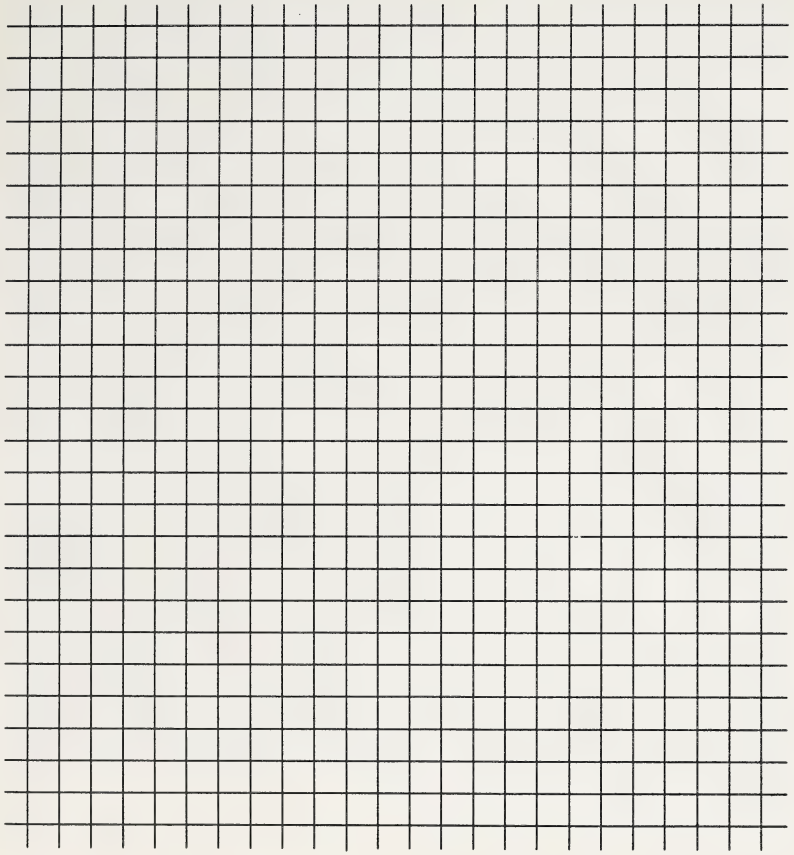


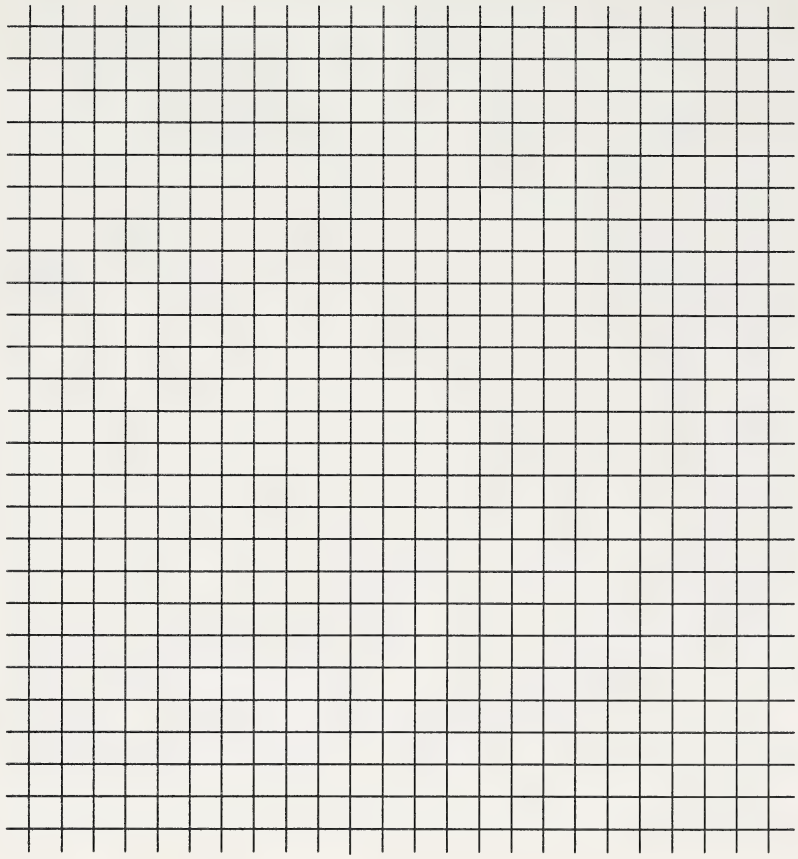


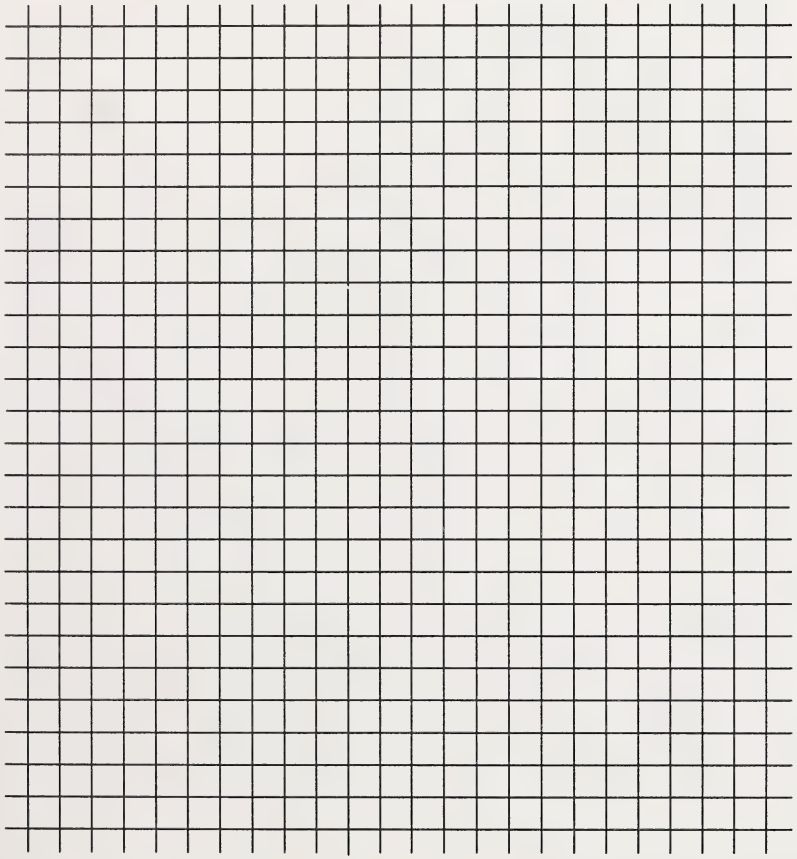


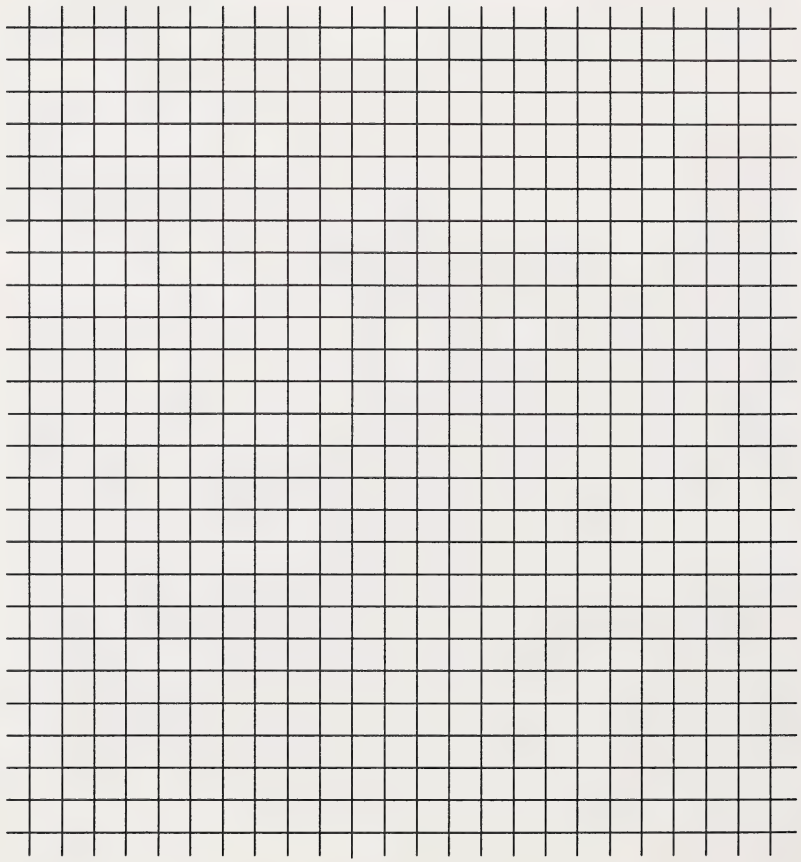


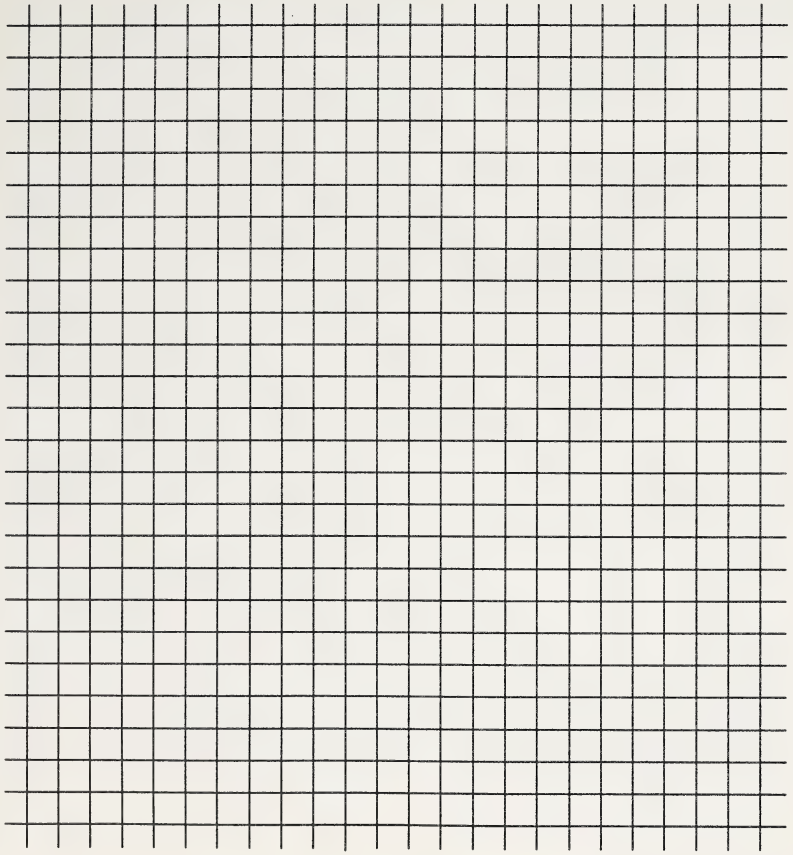


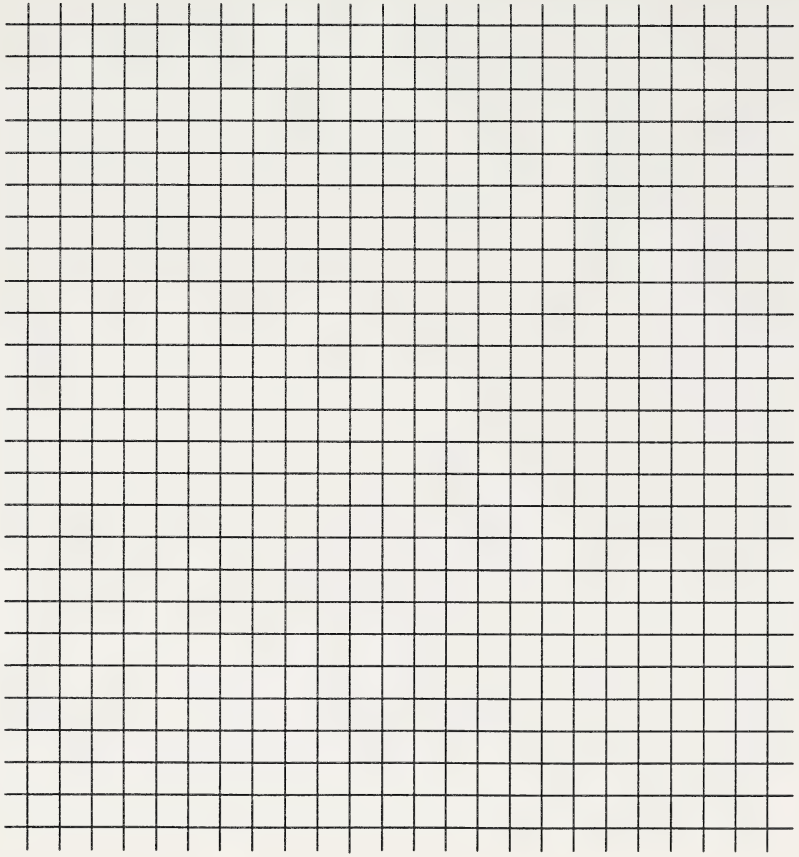


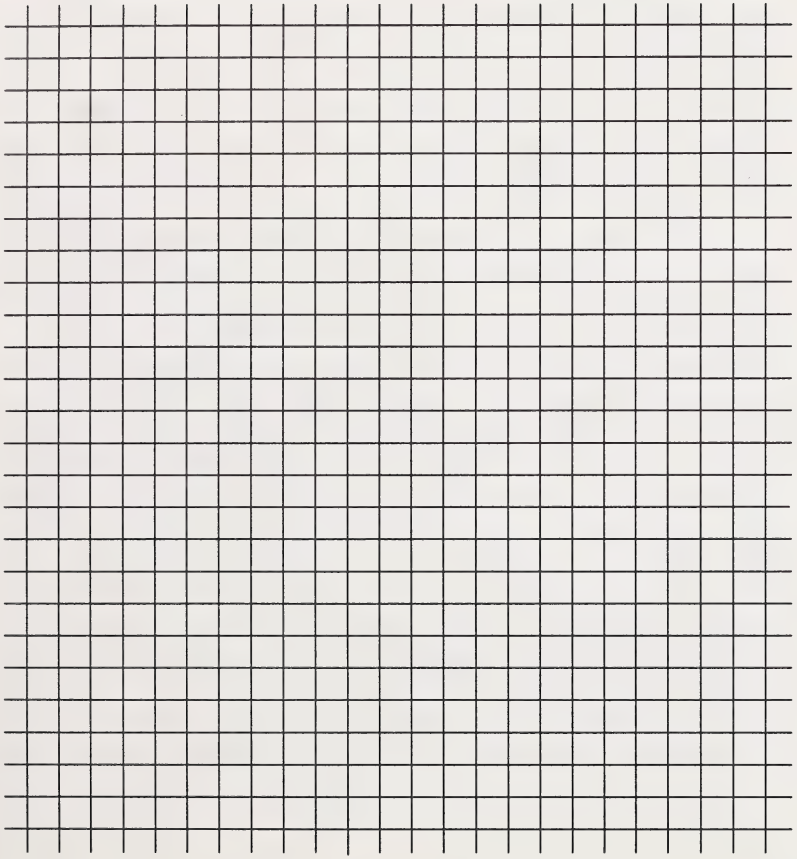


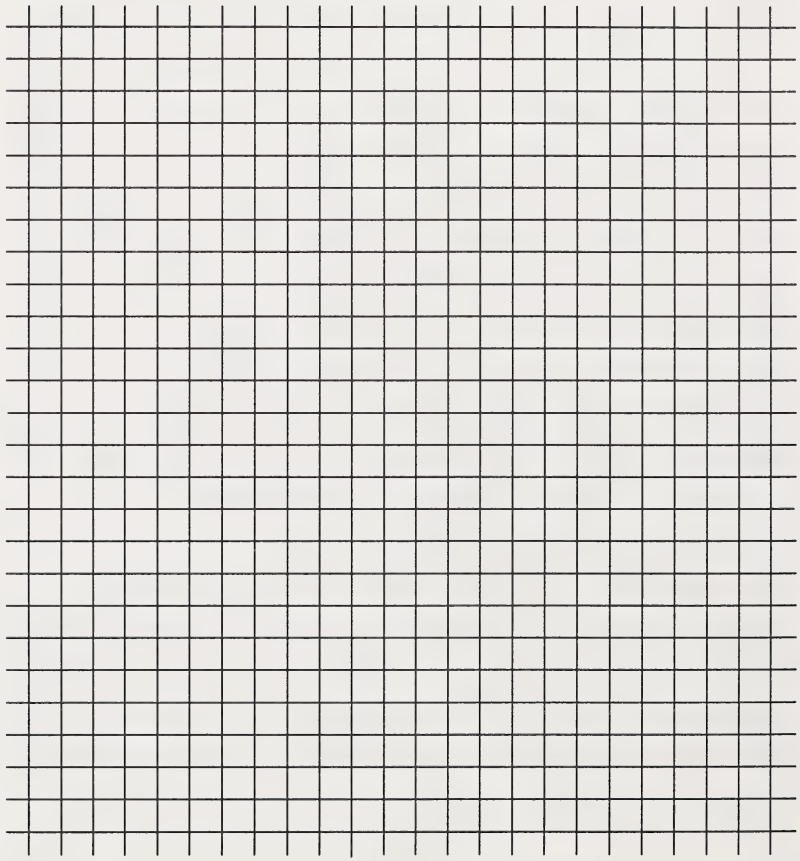


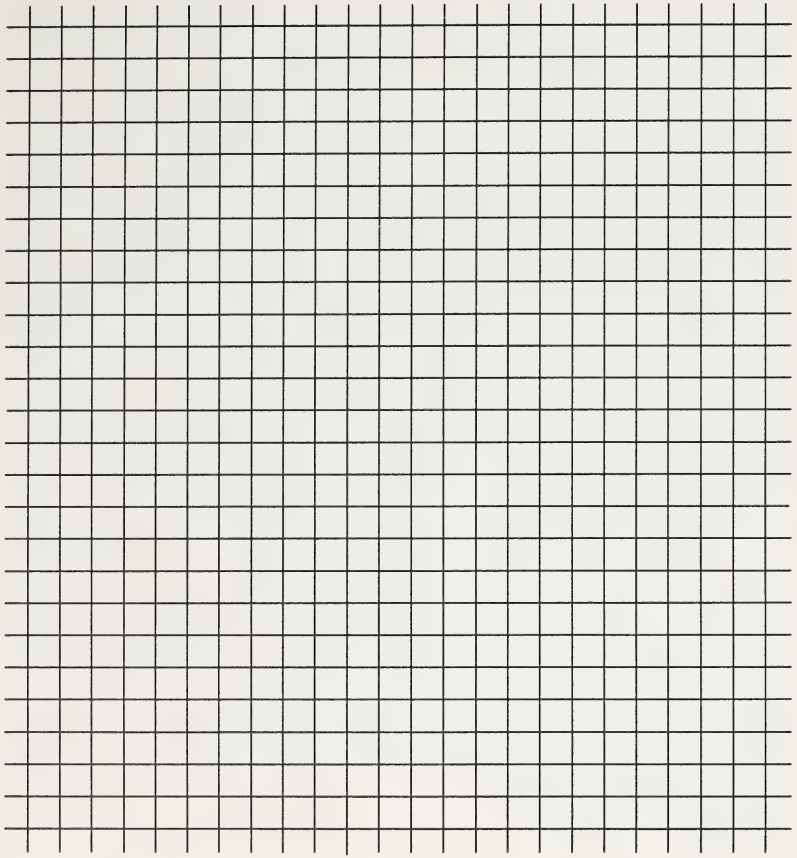


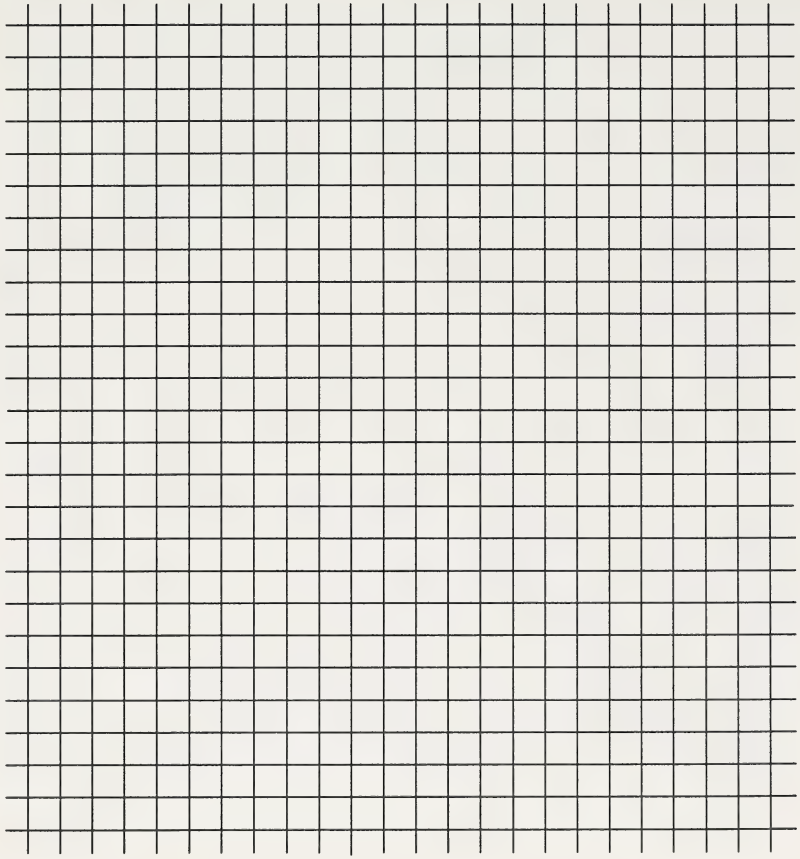


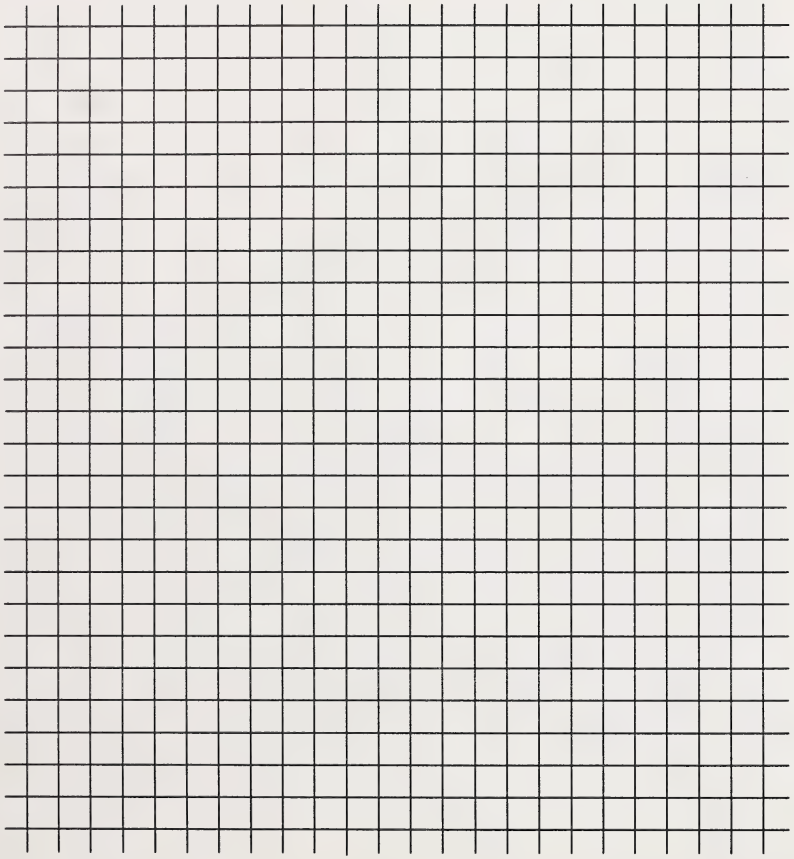


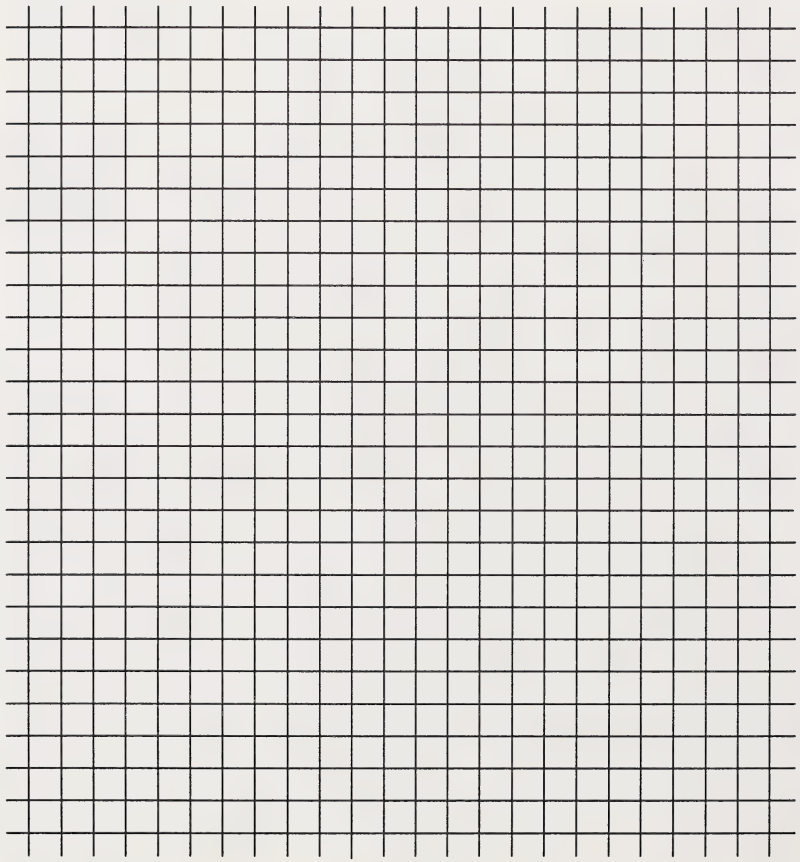


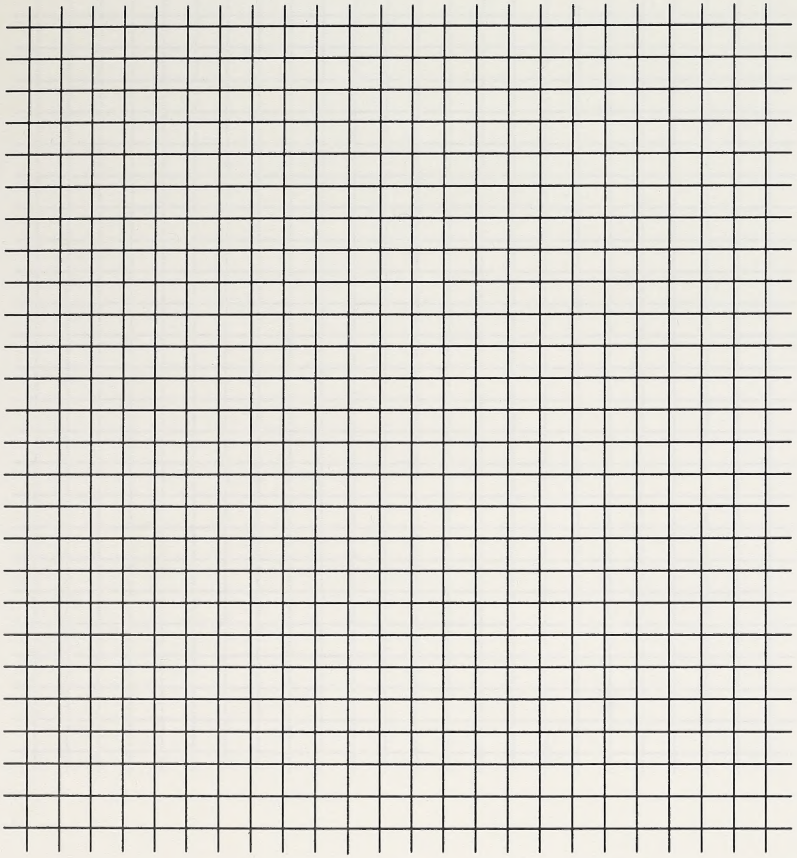


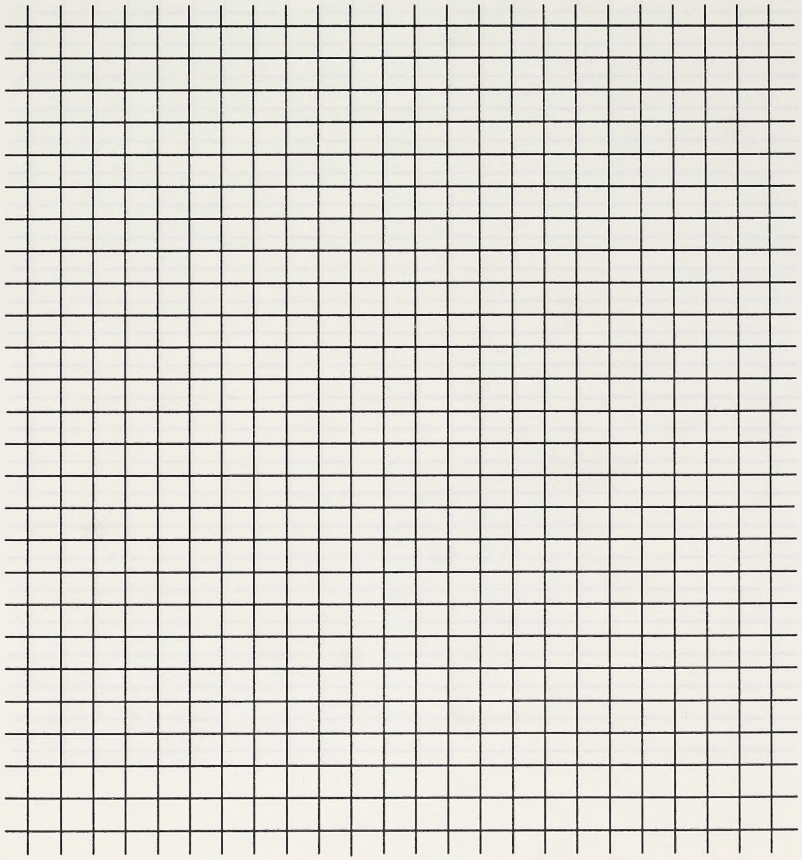


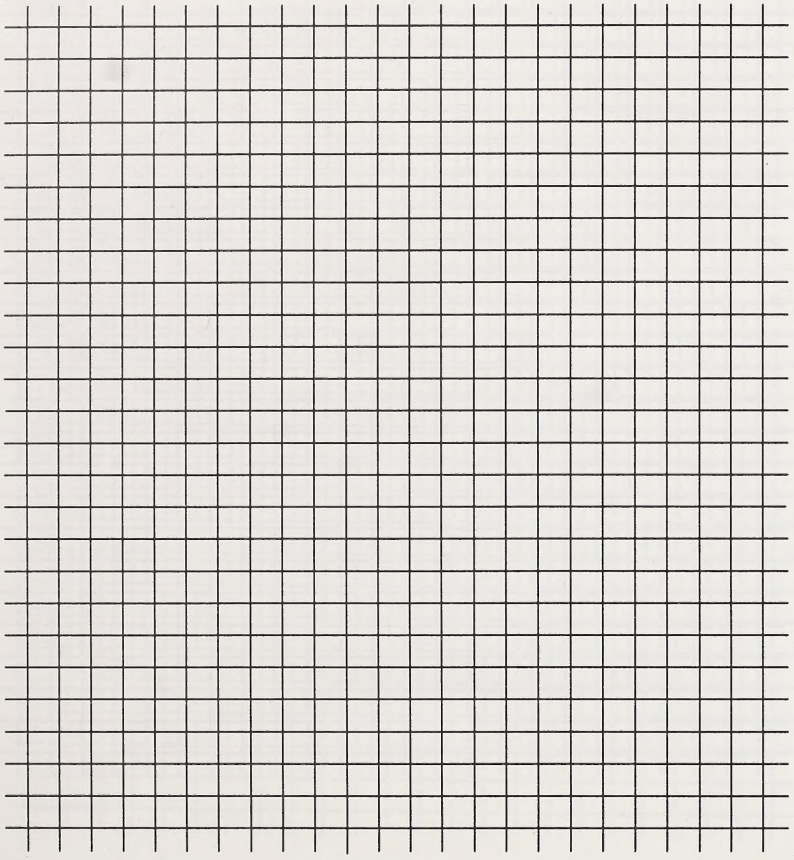


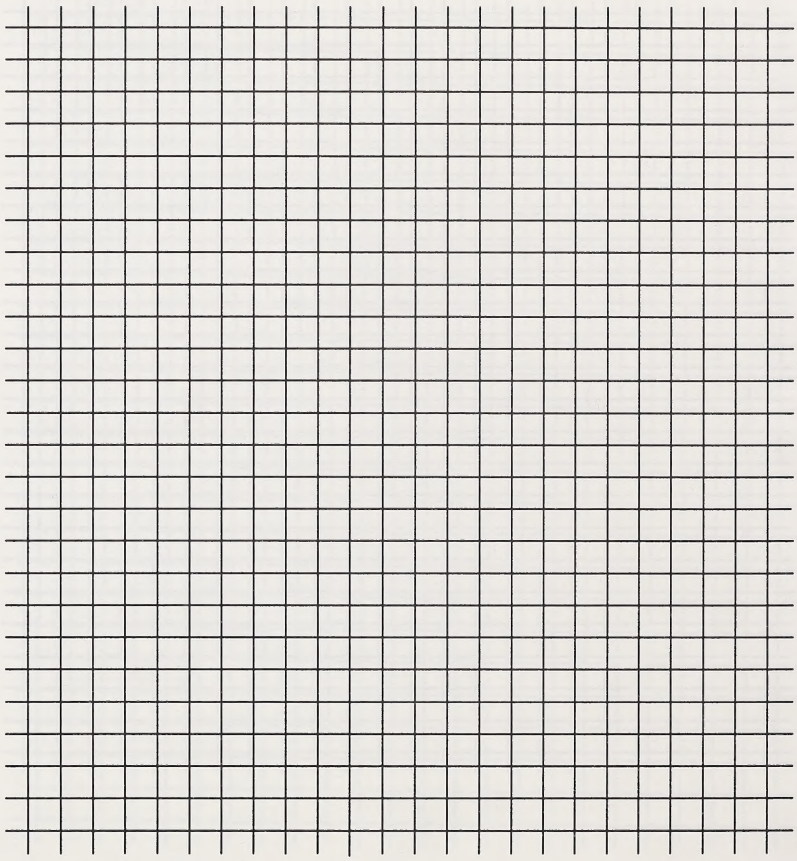












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